

Infinite sequences of almost Kaehler manifolds with high symmetry, their perturbations and pseudo holomorphic curves

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Abstract

We study analysis over infinite dimensional manifolds consisted by sequences of almost Kaehler manifolds. We develop moduli theory of pseudo holomorphic curves into such spaces with high symmetry. Many mechanisms of the standard moduli theory over finite dimensional spaces also work over these infinite dimensional spaces, which is based on a simple functional analytic framework.

Introduction

Global analysis of infinite dimensional spaces is one of the central theme to study in geometry. There would immediately occur several difficulties when we try to follow and apply any established methods over finite dimensional spaces, one of whose main reasons come from locally non compactness. Still the extensive developments have been achieved from the view points of the metric spaces, which essentially measure one dimensional objects.

In this paper, we study infinite dimensional geometry and analysis from two dimensional view points, and introduce a class of infinite dimensional geometric spaces consisted by sequences of embeddings by finite dimensional manifolds. We develop a foundation of analytic tool to perform some functional analysis under the conditions of high symmetry over such spaces.

Moduli theory of pseudo holomorphic curves turned out to be the very powerful tool and has become one of the central theme in symplectic geometry. From the view point of infinite dimensional geometry, it has led us to study *almost Kaehler sequences* $[(M_i, \omega_i, J_i)]$ which consist of families of embeddings by almost Kaehler manifolds:

$$[(M_i, \omega_i, J_i)] = (M_0, \omega_0, J_0) \subset (M_1, \omega_1, J_1) \subset \cdots \subset (M_i, \omega_i, J_i) \subset \cdots$$

In order to develop global analysis over these spaces, we introduce several set-ups of function spaces. Based on the analytic fields, we study maps into these infinite dimensional spaces. In particular we develop the moduli theory of pseudo holomorphic curves into such sequences from two dimensional spheres. Our construction provides with two main ingredients. One is Fredholm theory for the linearized maps, where it requires two conditions, closedness of the maps and finite dimensionality. The other is non linear analysis where it also requires two conditions, regularity of holomorphic maps and compactness of the moduli spaces. We discover that these properties also hold over the infinite dimensional spaces under the conditions of high symmetry.

As an application, these constructions allow us to calculate or estimate the capacity invariants over such infinite dimensional spaces. Capacity invariants have been introduced by Hofer and Zehnder with the axiom of the invariants over finite dimensional symplectic manifolds ([HZ]). Kuksin extended the construction of the capacity invariants over symplectic Hilbert spaces, and investigated invariance under the flow maps ([Ku], [B]). It turned out that they play an important role in relation with Hamiltonian PDE, whose phase spaces are infinite dimensional.

This paper is organized by three constructions:

(A) Formulation of a class of infinite dimensional spaces which allow us to analyze passing through infinite dimensional local charts. Infinite dimensional spaces behave quite flexibly under embeddings from each other. It allows us to introduce the *infinitesimal neighbourhoods* of such spaces, which are consisted by ‘convergent sequences’ of infinite dimensional spaces. These notions would provide us with rich classes of spaces which are able to apply our functional analytic methods.

(B) Construction of the moduli theory into such infinite dimensional spaces.

(C) Analysis of the capacity invariants over such spaces and their calculations. Combining with these constructions, we study stability of the capacity invariants under deformation in the infinitesimal neighbourhoods.

Let us explain more details of the contents.

We carefully introduce the differentiable functions over the infinite dimensional spaces, which should be able to be extended over their completion to the Hilbert manifolds.

We will verify that many mechanisms of the moduli theory of holomorphic curves work over almost Kaehler sequences with some symmetric conditions. Many of the infinite complex homogeneous spaces satisfy such symmetries.

$[(M_i, \omega_i, J_i)]$ is said to be a *symmetric almost Kaehler sequence*, if for any $k \geq 0$, there are some $l = l(k)$, families of almost Kaehler submanifolds $M_k \subset W_i \subset M_i$ for all $i \geq l$ with $W_l = M_l$ and isomorphisms:

$$P_i : \{(M, M_k), \omega, J\} \cong \{(M, M_k), \omega, J\}$$

with $M \equiv \cup_j M_j$, which preserve M_k and transform W_i to M_l as:

$$P_i : (W_i, \omega_i|_{W_i}, J_i|_{W_i}) \cong (M_l, \omega_l, J_l)$$

so that at any $p \in M_k$, $D : TM_k \oplus_{i \geq l} N_{i,k} \cong TM|_{M_k}$ give the uniform isomorphisms over M_k with respect to the complete local charts, where $N_{i,k}$ are isomorphic to the normal bundles of $TM_k \subset TM_l|_{M_k}$:

$$N_{i,k} = (P_i^{-1})_* [(\text{Ker } (\pi_k)_* \cap TM_l)|_{M_k}], \quad D_p = \text{id} \oplus (P_l)_* \oplus (P_{l+1})_* \oplus \dots$$

(Precisely see def 1.4.) Symmetric Almost Kaehler sequences satisfy the uniform isomorphisms which is one of the main property we need:

$$TM|_{M_k} \cong TM_k \oplus (N_{k,l} \otimes \mathbf{R}^\infty).$$

Our infinite dimensional geometry relies on the following functional analytic property:

Lemma 0.1 *Let $F : H \rightarrow H$ be a bounded operator with closed range and finite dimensional kernel. Then for any closed linear subspace $L \subset H$ and the Hilbert space tensor product $L \otimes W$ with another Hilbert space W , the image of the induced operators:*

$$(F \otimes 1)(L \otimes W) \subset H \otimes W$$

also have closed range.

Now let us introduce several functional spaces in order to develop moduli theory of holomorphic curves. Later on we fix two points $p_0, p_\infty \in M_0 \subset M$.

Let $E(J)_i, F_i \mapsto S^2 \times M_i$ be vector bundles whose fibers are respectively:

$$\begin{aligned} E(J)_i(z, m) &= \{\phi : T_z S^2 \mapsto T_m M_i : \text{anti complex linear}\}, \\ F_i(z, m) &= \{\phi : T_z S^2 \mapsto T_m M_i : \text{linear}\}. \end{aligned}$$

For fixed large $l \geq 1$, let $L_{l+1}^2(S^2, M_i)$ be the sets of L_{l+1}^2 maps from S^2 to M_i , and define the spaces of Sobolev maps:

$$\begin{aligned} \mathfrak{B}_i &\equiv \mathfrak{B}_i(\alpha) = \{u \in L_{l+1}^2(S^2, M_i) : [u] = \alpha, \\ &\int_{D(1)} u^*(\omega) = \frac{1}{2} \langle \omega, \alpha \rangle, \quad u(*) = p_* \in M_0, * \in \{0, \infty\}\}. \end{aligned}$$

Then we have two stratified Hilbert bundles over \mathfrak{B}_i :

$$\begin{aligned} \mathfrak{E}_i &= L_l^2(\mathfrak{B}_i^*(E(J)_i)) = \cup_{u \in \mathfrak{B}_i} \{u\} \times L_l^2(u^*(E(J)_i)), \\ \mathfrak{F}_i &= L_l^2(\mathfrak{B}_i^*(F_i)) = \cup_{u \in \mathfrak{B}_i} \{u\} \times L_l^2(u^*(F_i)). \end{aligned}$$

These spaces admit continuous S^1 actions.

The non linear Cauchy-Riemann operator is given as sections:

$$\bar{\partial}_J \in C^\infty(\mathfrak{E}_i \mapsto \mathfrak{B}_i), \quad \bar{\partial}_J(u) = Tu + J \circ Tu \circ i.$$

u is called a *holomorphic curve* if it satisfies the equation $\bar{\partial}_J(u) = 0$. The moduli space of holomorphic curves is defined by:

$$\mathfrak{M}(\alpha, M_i, J_i) = \{u \in C^\infty(S^2, M_i) \cap \mathfrak{B}_i(\alpha) : \bar{\partial}_J(u) = 0\}.$$

J is called regular, if the linealizations are onto for all $u \in \mathfrak{M}(\alpha, M_i, J_i)$ and all $i \geq 0$:

$$D\bar{\partial}_J(u) : T_u \mathfrak{B}_i \mapsto (\mathfrak{E}_i)_u$$

Let $u \in \mathfrak{B} \equiv \cup_i \mathfrak{B}_i$ and take an open neighbourhood $U(u) \subset \mathfrak{B}$. By introducing the Sobolev norms, one can make completion $U(u) \subset \hat{U}(u)$. Notice that elements in $\hat{U}(u)$ cannot be realized by maps into $M = \cup_i M_i$ in general. Let:

$$\bar{\partial}_J : \hat{U}(u) \mapsto \hat{\mathfrak{E}}|\hat{U}(u)$$

be the extension of the CR operator over the completion. The differential of the operator is not necessarily onto even if it is regular, where the range

may not be closed. An almost Kaehler sequence is said to be *strongly regular*, if the extensions are onto at all:

$$u \in \mathfrak{M}([(M_i, \omega_i, J_i)]) \equiv \cup_i \mathfrak{M}(\alpha, M_i, J_i).$$

This is the key property of moduli theory we develop in the infinite dimensional setting.

Theorem 0.1 *Let $[(M_i, \omega_i, J_i)]$ be a symmetric Kaehler sequence.*

(1) *Suppose it is regular and $\dim \cup_i \text{Ker } D_u \bar{\partial}_i = N$ is finite, then it is in fact strongly regular of index N .*

In particular $\mathfrak{M}([(M_i, \omega_i, J_i)])$ is a regular N dimensional S^1 free manifold.

(2) *Assume moreover it is isotropic, and each connected component of $\mathfrak{M}([(M_i, \omega_i, J_i)])$ is bounded. Then the equality holds:*

$$\mathfrak{M}([(M_i, \omega_i, J_i)]) = \mathfrak{M}(M_0, \omega_0, J_0).$$

In particular if it is minimal, then $\mathfrak{M}([(M_i, \omega_i, J_i)])$ is compact.

The second condition in (2) is satisfied when $N = 1$ (2.D).

Stability of geometric structures is one of the central theme in geometry and analysis of dynamics. Let us say that a family of almost Kaehler sequences *converges* :

$$\{[(M_i^l, \omega_i^l, J_i^l)]\}_{l \geq 1} \rightarrow [(M_i, \omega_i, J_i)]$$

as $l \rightarrow \infty$, if there are positive $\epsilon > 0$, subindices $\{k(i)\}_i$ and C^∞ and compatible embeddings:

$$I(i, l) : M_i^l \hookrightarrow M_{k(i)}$$

such that:

- (1) $\{I(i, l)(M_i^l)\}_{l \geq 1} \subset M_{k(i)}$ converges to M_i in $M_{k(i)}$ in C^∞ for each i ,
- (2) there are uniform extensions of almost complex $J(i, l)$ and symplectic $\omega(i, l)$ structures over the open ϵ tubular neighbourhoods:

$$I(i, l)(M_i^l) \subset U(i, l) \subset M \equiv \cup_i M_i$$

(3) there are compatible families of holomorphic maps:

$$\pi_k^l : (U(k, l), J(k, l)) \mapsto (M_k^l, J_k^l).$$

(Precisely see def 3.1.) Analysis of perturbations of moduli theory is to study asymptotic behaviour of structure of moduli spaces as $l \rightarrow \infty$. In light of this aspect, we introduce the *infinitesimal neighbourhoods* of $[(M_i, \omega_i, J_i)]$:

$$\mathfrak{N}([(M_i, \omega_i, J_i)])$$

which consist of the equivalent classes of the isomorphism classes of convergent families of almost Kaehler sequences, where:

$$\{[(M_i^l, \omega_i^l, J_i^l)]\}_{l \geq 1} \sim \{[(M_i^{l(h)}, \omega_i^{l(h)}, J_i^{l(h)})]\}_{h \geq 1}$$

for all infinite subindices.

We say that the infinitesimal neighbourhood is strongly regular, if for any element $[(M_i^l, \omega_i^l, J_i^l)] \in \mathfrak{N}([(M_i, \omega_i, J_i)])$, there is a large l_0 so that for all $l \geq l_0$, $[(M_i^l, \omega_i^l, J_i^l)]$ are all strongly regular.

In general regularity condition in moduli theory is not stable under convergence of almost Kaehler sequences. We verify that strong regularity condition overcomes this difficulty and controls analytic behaviour of holomorphic maps.

Suppose $[(M_i, \omega_i, J_i)]$ satisfies all the conditions in theorem 0.1: (1) regular, (2) minimal, (3) isotropically symmetric and (4) Kaehler sequence. (5) Moreover assume the moduli space $\mathfrak{M}([(M_i, \omega_i, J_i)])$ is 1-dimensional.

Then by theorem 0.1, the followings hold:

- (A) $\mathfrak{M}([(M_i, \omega_i, J_i)]) = \mathfrak{M}(M_0, \omega_0, J_0)$, and both are compact.
- (B) $[(M_i, \omega_i, J_i)]$ is strongly regular.

Let us introduce another geometric condition which can be applied to analysis over infinitesimal neighbourhoods. $[(M_i, \omega_i, J_i)]$ is *quasi transitive*, if for any $N > 0$, there is $k = k(N)$ so that automorphisms $A : (M, \omega, J) \cong (M, \omega, J)$ exist with $A(p_i) \in M_k$ for any $p_0, \dots, p_{N-1} \in M \equiv \cup_i M_i$ and all $0 \leq i \leq N - 1$.

Theorem 0.2 Assume moreover that (6) $[(M_i, \omega_i, J_i)]$ is quasi transitive. Then the followings hold:

(C) $\mathfrak{M}([(M_i, \omega_i, J_i)])$ is strongly regular.

(D) There are homeomorphisms:

$$\mathfrak{M}([(M_i, \omega_i, J_i)]) \cong \mathfrak{M}([(M_i^l, \omega_i^l, J_i^l)])$$

for any $[(M_i^l, \omega_i^l, J_i^l)] \in \mathfrak{M}([(M_i, \omega_i, J_i)])$ and all large l .

Here m stands for minimality of maps in π_2 .

In particular (D) implies that roughly speaking there are no ‘divergent sequences’ $u^l \in \mathfrak{M}([(M_i^l, \omega_i^l, J_i^l)])$ which approach to be holomorphic with respect to J , but they do not converge to any elements in $\mathfrak{M}([(M_i, \omega_i, J_i)])$.

We apply the analysis of moduli theory over almost Kaehler sequences to Hamiltonian dynamics.

Let (M, ω) be a finite dimensional symplectic manifold. A Hamiltonian function $f : M \mapsto [0, \infty)$ gives the Hamiltonian vector field X_f by the relation $df(\cdot) = \omega(X_f, \cdot)$. A pre-admissible function is called *admissible* if any non trivial periodic solutions have their periods larger than 1.

Suprimum of the widths $\sup f - \inf f$ among all the admissible functions is called the capacity invariant of (M, ω) , which contains deep information in Hamiltonian dynamics.

The capacity invariant $\text{cap}([(M_i, \omega_i, J_i)])$ over almost Kaehler sequences are straightforwardly defined by use of bounded Hamiltonians $f : M \rightarrow [0, \infty)$ over $M = \cup_i M_i$.

In order to study the capacity invariant over perturbations of almost Kaehler sequences, it turns out that the *asymptotic periodic solutions* of Hamiltonian arise quite naturally, which consist of a family of loops $x_i : [0, T_i] \rightarrow M_i$ with:

$$\sup_t |\dot{x}_i - X_{f_i}(x_i)|(t) \rightarrow 0, \quad i \rightarrow \infty.$$

Instead of periodic solutions, one can use asymptotic periodic solutions over Hamiltonians and define admissibility in a parallel way. Then one

obtains an analogue of capacity by use of such objects, which we call the *asymptotic capacity*:

$$\text{As-cap}([(M_i, \omega_i, J_i)]).$$

Conversely if we regard asymptotic periodic solutions as though ‘periodic solutions’ over elements of an infinitesimal neighbourhood, then it leads us to formulate the capacity invariant over the infinitesimal neighbourhoods:

$$C^m([(M_i, \omega, J_i)]) = \sup\{ \limsup_l \text{cap}([(M_i^l, \omega_i^l, J_i^l)]) : \\ [[(M_i^l, \omega_i^l, J_i^l)]] \in \mathfrak{N}^m([(M_i, \omega_i, J_i)]) \}$$

Notice that a priori estimates hold:

$$C^m \geq \text{cap} \geq \text{As-cap} \geq 0.$$

Theorem 0.3 *Let $[(M_i, \omega, J_i)]$ be a minimal, isotropically symmetric and quasi transitive Kaehler sequence, with a fixed minimal element $\alpha \in \pi_2(M)$.*

If the moduli space of holomorphic curves is non empty, regular, 1 dimensional and S^1 freely cobordant to non zero, then the estimate:

$$C^m([(M_i, \omega_i, J_i)]) \leq m$$

holds, where $m = \langle \omega, \alpha \rangle$.

Here we have the results of concrete calculations:

Proposition 0.1 (1) *Let Cap be As-cap or cap . Then:*

$$C^m([\mathbf{CP}^i]) = \text{Cap}([\mathbf{CP}^i]) = \text{Cap}([D^{2i}]) = \pi.$$

$$(2) \ C([(T^{4i}, \omega, J)]) = \infty.$$

Furthur research directions: Let us describe possible developments for future, which we partly study in this paper.

One of the important development is study of the displacement energies, since our techniques in this paper can be applied for $X \times \lambda \mathbf{CP}^1$ if X is the

case with the additional condition of projective or π_2 rank 1, where $\lambda \mathbf{CP}^1$ is \mathbf{CP}^1 equipped with the rescaled Fubini-Study form by $\lambda > 0$ (see [LM]). Let $M = \cup_i M_i$ and \bar{M} be its completion to the Hilbert manifold. Let us consider a bounded Hamiltonian $H : M \rightarrow [0, \infty)$ and $D_H : \bar{M} \cong \bar{M}$ be the Hamiltonian vector field (5.C).

Let us say that an open subset $U \subset M$ is *displaceable*, if there is a bounded Hamiltonian H as above so that $D_H(\bar{U}) \cap \bar{U} = \emptyset$ hold in \bar{M} , where \bar{U} is the closure of U in \bar{M} . We say that H displaces U , and we expect the energy estimates $\sup_{m \in M} H(m) \geq \text{As-cap}(U)$. For $X \subset \mathbf{CP}^\infty$, let $U(X) \subset \mathbf{CP}^\infty$ be the maximal open neighbourhoods of X such that $U(X)$ are displaceable. Estimates of capacity values of $U(X)$ would be of particular interests for us, which might measure ‘symplectic complexity’ of X . Below we introduce two categories:

(A) *Projective varieties*: (Finite dimensional) projective varieties admit embeddings into \mathbf{CP}^n for some n . Let us allow to address two types of questions which are of interests for us.

Let us consider the moduli space of, say $K3$ surface, and take a non projective (but always Kaehler) variety X which admit projective $K3$ surfaces $X_i \subset \mathbf{CP}^{n_i}$ converging to X in the moduli spaces. One can find some X such that these projective varieties must satisfy $n_i \rightarrow \infty$. Then we ask what are the asymptotic behaviour of values of $\{\text{As-cap}(U(X_i))\}_i$.

Let X be a projective variety (possibly infinite dimension), and \mathfrak{F} be the class of projective embeddings. There are canonical embeddings such as Veronese and Plücker embeddings. One may wonder whether positivity $\text{As-cap}(U(X)) > 0$ might hold for both $X = Pl_r(Gr_r), v_m(\mathbf{CP}^\infty) \subset \mathbf{CP}^\infty$.

Let us fix $0 < \delta < 1$ and consider δ balls:

$$B^{2n}(\delta) = \{(z_1, \dots, z_n) : \sum_{i=1}^n |z_i|^2 < \delta^2\} \subset \mathbf{C}^n.$$

equipped with the standard symplectic structure. They are embedded into \mathbf{CP}^n by:

$$(z_1, \dots, z_n) \in B^{2n}(\delta) \hookrightarrow [\sqrt{1 - |z|^2}, z_1, \dots, z_n]$$

which preserve the symplectic structure compatibly with the Fubini-Study forms. It induces the infinite dimensional embedding $I_2 : B^{2\infty}(\delta) \subset \mathbf{CP}^\infty$.

Let $I_1 : B^{2\infty}(\delta) \hookrightarrow B^{2\infty}(\delta)$ be $(z_1, \dots, z_n, \dots) \rightarrow (0, z_1, \dots, z_n, \dots)$, and consider the composition:

$$S \equiv I_2 \circ I_1 : B^{2\infty}(\delta) \hookrightarrow \mathbf{CP}^\infty$$

Lemma 0.2 *For $0 < \delta < 1$, there are displaceable $U(S(B^{2\infty}(\delta)))$ so that positivity holds:*

$$As\text{-}cap(U(S(B^{2\infty}(\delta)))) > 0.$$

Proof: Let us choose $0 < \delta < \delta' < 1$, and consider the embeddings $B^{2\infty}(\delta) \hookrightarrow B^\infty(\delta')$ by $(z_1, \dots, z_n, \dots) \rightarrow (0, z_1, \dots, z_n, \dots)$.

Let $f : [0, \infty) \rightarrow [0, 1]$ be a smooth function with $f|_{[0, \delta^2]} \equiv \alpha > 0$ and $f|_{[(\delta')^2, \infty)} \equiv 0$, where $\alpha > 0$ is sufficiently small. Let us put $z_0 = x_0 + \sqrt{-1}y_0$ and $\bar{z} = (z_1, z_2, \dots)$. Then we have the bounded Hamiltonian:

$$F : B^{2\infty}(\delta') \rightarrow \mathbf{R}, \quad F(z_0, \bar{z}) = f(|\bar{z}|^2)x_0.$$

Notice that it extends over \mathbf{CP}^∞ by letting 0 outside $B^{2\infty}(\delta')$. The Hamiltonian vector field is given by the form $X_F = (-\alpha, 0, *)$ on small neighbourhood of $S(B^{2\infty}(\delta))$. In particular the first coordinate of $D_F(S(B^{2\infty}(\delta)))$ must be uniformly away from 0. This completes the proof.

Let us list two important embeddings into infinite dimensional projective spaces:

Bergman (pseudo) metrics: Let X be a finite dimensional complex manifold, and W be the space of L^2 -holomorphic n forms over X . The canonical inner product over W gives it a Hilbert space structure, which gives rise to the Bergman kernel form. When it is pointwisely positive, then the map $I : X \mapsto \mathbf{CP}(W^*)$ is induced. When it is immersion then the Bergman metric is equipped over X by pull-back of the Fubini-Study.

Rational dynamics: In [K3], we have studied analytic behaviour of rational dynamics from the view point of scaling limits. The simplest case of infinite algebraic varieties will be the shift:

$$S : \mathbf{CP}^\infty \hookrightarrow \mathbf{CP}^\infty, \quad [z_0, z_1, \dots] \rightarrow [0, z_0, z_1, \dots]$$

whose image is given by the homogeneous polynomial $F(z_0, z_1, \dots) = z_0$.

Let $f(z_0, \dots, z_{n-1})$ be a rational function and consider the iteration dynamics given by the rule $z_N = f(z_{N-n}, \dots, z_{N-1})$ with the initial values $(z_0, \dots, z_{n-1}) \in \mathbf{C}^n$. Let us say that the orbits $\{z_N\}_N$ are affine, if $(z_0, z_1, \dots) \in \mathbf{C}^\infty \subset \mathbf{CP}^\infty$ by the embedding $(z_0, z_1, \dots) \rightarrow [1, z_0, z_1, \dots]$, and they are rational if $[1, z_0, z_1, \dots]$ determine points in \mathbf{CP}^∞ . Now we have X as all the set of rational orbits.

f gives the *recursive* dynamics, if there exists some M so that for any initial values, the corresponding orbits satisfy the equalities $z_{N+M} = z_N$ hold for all $N \geq 0$. When f is recursive, non affine orbits are parametrized by algebraic sets in \mathbf{C}^n .

Example: Let us consider $f(z_0, z_1) = z_0^{-1}(1 + z_1)$. It is well-known that this gives the recursive dynamics of period 5. Notice that any affine orbits $\{z_N\}_N$ must satisfy $z_i \neq 0$ for all i , which is equivalent to $z_i \neq 0$ for $0 \leq i \leq 4$. By straight forward computations, the non affine set is given by $V_s = \{(z_0, z_1) : z_0 z_1 = 0\}$.

(B) *Complex vector bundles:* Let $[\mathbf{Gr}_{n,i}]$ be the complex Grassmannians and put $\mathbf{Gr}_n = \cup_i \mathbf{Gr}_{n,i}$. Let M be a complex manifold of dimension $0 \leq n < \infty$, and consider the classifying maps $f : M \rightarrow \mathbf{Gr}_n$ for TM with $X = f(M) \subset \mathbf{Gr}_n$. By modifying our functional spaces by change of dimension of fixing points as in defining the quantum cohomology, one will be able to apply our techniques in this paper to the infinite Grassmannians.

1 Almost Kaehler sequences

1.A Function spaces over local charts: For positive $\epsilon > 0$, let $D^{2k}(\epsilon) \subset \mathbf{R}^{2k}$ be ϵ ball with the center 0. We denote the $2i$ dimensional ϵ cube $D(i) = D^2(\epsilon) \times \dots \times D^2(\epsilon)$ by multiplications of $D^2(\epsilon)$ by i times. There are canonical embeddings $D_i = D(i) \times \{0\} \subset D_{i+1}$ for all $i \geq 1$. Let us put the infinite dimensional cube and disk by:

$$D(\infty) \equiv \cup_{i \geq 1} D_i, \quad D(\epsilon) \equiv \cup_{k \geq 1} D^{2k}(\epsilon) \subset \mathbf{R}^\infty$$

respectively. Notice $\text{diam } D(\infty) = \infty$.

1.A.1 Hilbert completion: Let H be the separable Hilbert space which

is obtained by the completion of \mathbf{R}^∞ with the standard metric on it. For $p = (p_0, p_1, \dots) \in \mathbf{R}^\infty$, let us denote by $D(\infty)(p) \equiv D(\infty) + p$ and $D(\epsilon)(p) \equiv D(\epsilon) + p$ as the infinite dimensional cube and disk with the center p respectively. We denote the metric completion by:

$$\bar{D}(\infty)(p), \quad \bar{D}(\epsilon)(p) \subset H.$$

By a neighbourhood or an open subset of $p \in B \subset \mathbf{R}^\infty$, we mean that B contains some $\delta > 0$ disk with the center p , with respect to the metric.

Let $p \in B \subset \mathbf{R}^\infty$ be an open subset, and denote its closure by $\bar{B} \subset H$. Let us consider a smooth and bounded function $f : B \mapsto \mathbf{R}$. We will regard the derivatives of f at p as the linear operators:

$$\nabla f : T_p \mathbf{R}^\infty \equiv \cup_{k \geq 1} T_p \mathbf{R}^{2k} \mapsto \mathbf{R}, \quad \nabla^2 f : (T_p \mathbf{R}^\infty)^{\otimes 2} \mapsto \mathbf{R}, \text{ etc.}$$

where $\nabla^2(f)(v, w) = \frac{\partial^2}{\partial s \partial t} f(p + sv + tw)|_{s=t=0}$ just for clarity.

Let us denote the operator norm by $|\nabla^l f|(p)$, if it extends to the bounded linear functional:

$$\nabla^l f : (T_p H)^{\otimes l} \mapsto \mathbf{R}.$$

Definition 1.1 *We will say that f is of completely C^k bounded geometry at $p = (p_0, p_1, \dots) \in B$, if:*

- (1) $f|_B$ extends to a continuous function on some neighbourhood of p in $\bar{B}(p)$ and
- (2) each differential $\nabla^l f : T_p H \mapsto \mathbf{R}$ exists continuously on some neighbourhood of p in $\bar{B}(p)$ for all $0 \leq l \leq k$ (so $|\nabla^l f|(p) < \infty$ hold)

We say f is of completely C^k bounded geometry, if it is at any point $p \in B$ satisfying uniformity:

$$\|f\|^2 C^k(B) \equiv \text{Sup}_{p \in B} \sum_{0 \leq l \leq k} |\nabla^l f|^2(p) < \infty.$$

Just completely bounded geometry implies C^∞ .

A pointwise operator D on functions over B is of completely bounded geometry, if D extends to a smooth operator $D(p)$ over functions of completely bounded geometry over $\bar{B}(p)$ at each $p \in B$. Namely the followings are satisfied:

(1) There is a constant C with:

$$D : C^0(B) \mapsto C^0(B), \quad |Df|(p) \leq C|f|(p), \quad p \in B.$$

We denote its extended and pointwise operator norm as $|D(p)|$.

(2) For all k , the following norms are all finite:

$$||D||^2 C^k(B) \equiv \text{Sup}_{p \in B} \sum_{0 \leq l \leq k} |\nabla^l D|^2(p) < \infty.$$

D gives a *complete isomorphism*, if it is of completely of bounded geometry. Moreover there are constants $0 < c < c'$ so that the uniform bounds hold for each $p \in B$:

$$c \leq ||D_p|| \leq c'.$$

For pointwisely bilinear forms, one has the parallel notion of *complete nondegeneracy*.

Later we will always treat almost Kaehler sequences whose almost complex structures, symplectic structures or the induced Riemannian metrics are all completely nondegenerate (1.B).

Example 1.1: Let $D^2 \subset \mathbf{R}^2$ be the standard ball with the center 0, and consider smooth functions $g, h : D^2 \mapsto [0, 1]$ where:

$$g(x) = \exp\left(-\frac{|x|^2}{1 - |x|^2}\right), \quad h(x) = \exp\left(-\frac{|x|}{1 - |x|}\right).$$

Let us prepare infinite numbers of the same g and h , and in order to distinguish these, let us assign indices as $g_i, h_i : D_i^2 \mapsto [0, 1]$. Then we consider functions over $D(\infty) = D_0^2 \times D_1^2 \times \dots$:

$$G = g_0 g_1 g_2 \dots, \quad H = h_0 h_1 h_2 \dots$$

by the pointwise multiplication. Both G and H are smooth on $D(\infty)$. They have the following properties:

(1) G is of completely bounded geometry on the unit balls with the center zero, and (2) H is not at any points.

H is not even continuous on $\bar{D}(\infty)$. In fact let us choose families of points

$\{p(l) = (p_0(l), p_1(l), \dots)\} \subset \bar{D}(\infty) \setminus D(\infty)$ with:

$$\begin{aligned} |p(l)|^2 L^2 &\equiv \sum_{i=0}^{\infty} |p_i(l)|^2 \rightarrow 0, \quad l \rightarrow \infty, \\ |p(l)| L^1 &\equiv \sum_{i=0}^{\infty} |p_i(l)| = \infty. \end{aligned}$$

Then clearly $H(p(l)) = 0$, but $H(0) = 1$.

Let $f : B \rightarrow \mathbf{R}$ be of completely C^1 bounded geometry. Then the one form $df = \sum_{i=1}^{\infty} \frac{\partial f}{\partial x_i} dx_i$ can be interpreted as a continuous map:

$$df : B \rightarrow H.$$

Then its higher derivatives give the functionals:

$$\nabla^l df : T\bar{B}^{\otimes l} \rightarrow H.$$

We define df is of completely bounded geometry, if f is of completely C^1 bounded geometry, and its higher derivatives $\nabla^l df$ give continuous maps with respect to the operator norms for all $l \geq 0$.

Lemma 1.1 *Let $D : C^0(B, H) \mapsto C^0(B, H)$ be a pointwise linear functional, and assume it gives a complete isomorphism. Then its inverse also gives a complete isomorphism.*

Proof: By the assumption, the inverse:

$$D^{-1} : C^0(B, H) \mapsto C^0(B, H)$$

satisfy the equalities:

$$\nabla(D^{-1}) = -D^{-1} \circ \nabla D \circ D^{-1}.$$

It is immediate to see that D^{-1} is also of completely bounded geometry.

This completes the proof.

1.A.2 Local charts: Let $(M_0, g_0) \subset (M_1, g_1) \subset \dots$ be embeddings of Riemannian manifolds with $\dim M_i = 2d_i$, where we assume the compatibility condition $g_{i+1}|_{M_i} = g_i$. We will denote such families by $[(M_i, g_i)]$. For $p, q \in M_i$, let us denote their distance in $M \equiv \cup_i M_i$ by:

$$d(p, q) \equiv \inf_{j \geq i} d_j(p, q).$$

We denote ϵ tubular neighbourhood of M_i by $U_\epsilon(M_i) \subset M$:

$$U_\epsilon(M_i) = \{m \in M : d(m, M_i) < \epsilon\}$$

Let us recall $D^{2i}(\epsilon) \subset \mathbf{R}^{2i}$ and $\bar{D}(p)$ be as in 1.A.1. Below we regard the Riemannian metrics $g = \{g_i\}_i$ as the pointwise operator over the local charts $T_p D \equiv \cup_i T_p D^{2i}(\epsilon)$, $p \in D$. If g is of completely bounded geometry at $p \in D$, then one can extend it to an operator on the Hilbert families $T_p \bar{D}(p)$.

Definition 1.2 *The Riemannian family $\{g_i\}_i$ is uniformly bounded, if the following conditions are satisfied. There exists a positive $\epsilon > 0$ such that:*

(1) *Every point $p \in M \equiv \cup_i M_i$ admits a stratified local chart:*

$$D^{2i}(\epsilon) \subset D^{2(i+1)}(\epsilon) \subset \cdots \subset \mathbf{R}^{2\infty}, \quad \varphi(p) : D(\epsilon) \equiv \cup_i D^{2i}(\epsilon) \hookrightarrow \cup_i M_i, \\ \varphi(p)_i \equiv \varphi(p)|_{D^{2d_i}(\epsilon)} \mapsto M_i, \quad \varphi(p)(0) = p.$$

(2) *With respect to $\varphi(p)$ as above, the induced Riemannian metrics $\{g_i\}_i$ are uniformly of completely bounded geometry. This means that for any $l > 0$, there are constants $C(l) \geq 0$ independent of p so that the estimates hold:*

$$\text{Sup}_{p \in M} \text{Sup}_{m \in \cup_i D^{2i}(\epsilon)} \sum_{0 \leq k \leq l} |\nabla^k(\varphi(p)^* g)|(m) \leq C(l) \quad (*)$$

(3) *There is an increasing and proper function $h : (0, \infty) \rightarrow (0, \infty)$ so that for any i , any pairs of points $p, q \in M_i$ satisfy the uniformly bounded distance property:*

$$d(p, q) \geq h(d_i(p, q))$$

where d_i and d are the distances on M_i and $M = \cup_i M_i$ respectively.

We will say that the stratified local charts as above are *complete local charts*. Also the above family $\{(p, \varphi(p))\}$ will be called a *uniformly bounded covering*. Later on uniform implies independence of choice of points as above.

Let $f : M = \cup_i M_i \mapsto \mathbf{R}$ be a bounded smooth function and:

$$\varphi(p)^*(f) : D(\epsilon) \rightarrow \mathbf{R}$$

be the induced functions with respect to the uniformly bounded covering. We say that f is of *completely bounded geometry*, if all $\varphi(p)^*(f)$ are uniformly of completely bounded geometry as:

$$\|f\|C^k(M) \equiv \sup_{p \in M} \|\varphi(p)^*(f)\|C^k(D(\epsilon)) \leq C_k$$

for all $k = 0, 1, 2, \dots$ and some constants C_k independent of $p \in M$.

Lemma 1.2 *Let $[(M_i, g_i)]$ be Riemannian embeddings as above with a uniformly bounded covering $\{(p, \varphi(p))\}$ with $\epsilon > 0$. Then $\exp_p : \bar{D}(\epsilon') \mapsto \bar{D}(\epsilon)$ exists and is smooth, with respect to the induced Riemannian metrics $\varphi(p)^*(g)$, where $\epsilon > \epsilon' > 0$ and we regard $D(\epsilon') \subset T_0 \bar{D}(\epsilon)$.*

For a proof, see [Kl] (p57, p72). Notice that the geodesic coordinate does not preserve the stratifications.

Let $f_n, g : M \rightarrow \mathbf{R}$ be a smooth and bounded family of functions for $n = 0, 1, 2, \dots$. We say that $\{f_n\}_n$ converges weakly to g in C^l , if the restrictions converge in C^l for all $k = 0, 1, 2, \dots$:

$$f_n|_{M_k} \rightarrow g|_{M_k}.$$

Lemma 1.3 *Let $[(M_i, g_i)]$ be a uniformly bounded Riemannian family such that each M_k is compact. Let $f_n : M \rightarrow \mathbf{R}$ be a family of smooth functions of completely bounded geometry for $n = 0, 1, 2, \dots$*

Suppose C^{l+1} norms are uniformly bounded:

$$\|f_n\|C^{l+1}(M) \leq \text{Const}(l).$$

Then a subsequence f_{n_j} weakly converges in C^l to a smooth function $g : M \rightarrow \mathbf{R}$ of completely bounded geometry.

Proof: By the condition, the restrictions $\{f_n|_{M_k}\}_n$ satisfy uniformity of C^{l+1} norms $\|f_n\|C^{l+1}(M_k) \leq C(l)$.

It follows from Rellich lemma that there is some C^l function $g_1 : M_1 \rightarrow \mathbf{R}$ so that a subsequence $\{f_{n(i)}|_{M_1}\}_i$ converges to g in $C^l(M_1)$.

By the same argument, there is some C^l function $g_2 : M_2 \rightarrow \mathbf{R}$ so that a subsequence $\{f_{n(i,2)}|_{M_2}\}_i$ converges to g_2 in $C^l(M_2)$ for another subsequence $\{n(i,2)\}_i \subset \{n(i)\}_i$. Clearly $g_2|_{M_1} = g_1$ holds.

By choosing subsequences successively, $\{f_{n(i,k)}|M_k\}_i$ converge to some C^l function $g_k : M_k \rightarrow \mathbf{R}$, with $g_k|_{M_{k-1}} = g_{k-1}$. These satisfy uniformity of C^l norms $\|g_k\|_{C^l(M_k)} \leq c < \infty$.

Let $g : M \rightarrow \mathbf{R}$ be a smooth and bounded function defined by $g|_{M_k} \equiv g_k$. Then the subsequence $\{f_{n(i,i)}\}_i$ converge weakly to g in C^l .

This completes the proof.

1.B Almost Kaehler sequence: Let (M, ω, J) be a finite dimensional symplectic manifold equipped with a compatible almost complex structure. Namely $g(\cdot, \cdot) = \omega(\cdot, J\cdot)$ gives a Riemannian metric on M . Such a manifold is called an *almost Kaehler manifold*.

Let $(M_0, \omega_0, J_0) \subset (M_1, \omega_1, J_1) \subset \dots \subset (M_i, \omega_i, J_i) \subset \dots$ be infinite embeddings of almost Kaehler manifolds. If one denotes the inclusion $I(i) : M_i \hookrightarrow M_{i+1}$, then the above implies $\{I(i)\}_i$ gives a family of holomorphic embeddings, $J_{i+1} \circ I(i)_* = I(i)_* \circ J_i$, and the symplectic forms $I(i)^*(\omega_{i+1}) = \omega_i$ are the restrictions.

Suppose $\dim M_k = 2d_k$, and $U_\epsilon(M_i) \subset M \equiv \cup_j M_j$ be ϵ tubular neighbourhoods of M_i . Let:

$$\pi'_k : \cup_i D^{2i}(\epsilon) \mapsto D^{2d_k}(\epsilon)$$

be the standard projections.

Definition 1.3 *An almost Kaehler sequence consists of a family of embeddings by almost Kaehler manifolds:*

$$[(M_i, \omega_i, J_i)] = (M_0, \omega_0, J_0) \subset (M_1, \omega_1, J_1) \subset \dots \subset (M_i, \omega_i, J_i) \subset \dots$$

and a positive $\epsilon > 0$ so that any points $p \in M \equiv \cup_i M_i$ admit ϵ uniformly bounded coverings $\{(p, \varphi(p))\}$ which satisfy the followings:

- (1) $\omega|_{\cup_i D^{2i}(\epsilon)}$ and $J|_{\cup_i D^{2i}(\epsilon)}$ are of completely bounded geometry.
- (2) The induced symplectic form satisfies:

$$\varphi(p)^*(\omega) = \frac{\sqrt{-1}}{2} \sum_{i=0}^{\infty} dw_i \wedge d\bar{w}_i \quad \text{at } p$$

where (w_1, \dots, w_i) are the coordinates on $D^{2i}(\epsilon) \subset \mathbf{C}^i$.

(3) *There are families of holomorphic maps:*

$$\pi_k : U_\epsilon(M_k) \mapsto M_k$$

which satisfy the following properties:

$$\pi_k|_{M_k} = id, \quad \pi_k(\varphi(p)(m)) = \varphi(p)(\pi'_k(m)) \quad \text{for all } m \in U_\epsilon(M_k).$$

Uniformly bounded coverings which satisfy the properties (1) (2) (3) above, are called ϵ *complete almost Kaehler charts*.

An almost Kaehler data $\{(\omega_i, J_i)\}$ gives a uniformly bounded and compatible family of Riemannian metrics on $\{M_i\}_i$. Notice that the equalities $\langle v, u \rangle = \langle \pi_k(v), u \rangle$ hold for $u \in T_p M_k$ and $v \in T_p U_\epsilon(M_k)$ with respect to the induced Riemannian metric.

Later on, we fix a uniformly bounded covering by ϵ complete almost Kaehler charts.

We will say that $[(M_i, \omega_i, J_i)]$ is a *Kaehler sequence*, if it is an almost Kaehler sequence consisted by a uniformly bounded covering by holomorphic complete Kaehler charts $\varphi(p)$ at all points p , where we equip with the standard complex structure on $\cup_i D^{2i}(\epsilon)$ (see [GH] p107).

Let $f : M = \cup_k M_k \rightarrow \mathbf{R}$ be a smooth bounded function over an almost Kaehler sequence. We will say that f is a *bounded Hamiltonian function*, if it is of completely bounded geometry.

Let (M, ω) be a finite dimensional symplectic manifold. The following facts are well known ([G1]):

- (1) there exist compatible almost complex structures, and
- (2) the space of compatible almost complex structures is contractible.

In the infinite dimensional situation, the condition (1) depends on the spaces, but the same thing holds for (2) for a fixed uniformly bounded covering.

Lemma 1.4 *Let $[(M_i, \omega_i)]$ be a symplectic sequence. Suppose there exists a family of compatible almost complex structures $\{J_i\}_i$ so that $[(M_i, \omega_i, J_i)]$*

consists of an almost Kaehler sequence with respect to a uniformly bounded covering $\{(p, \varphi(p))\}$. Then the space of such family:

$$\mathfrak{J}([(M_i, \omega_i)]) = \{ \{J_i\}_i : [(M_i, \omega_i, J_i)] : \\ \text{almost Kaehler sequence with respect to } \{(p, \varphi(p))\} \}$$

is contractible.

Proof : We follow a well known argument in the finite dimensional case.

Let us choose a reference family of almost complex structures $\{J_i^0\}_i$. Take another one $\{J_i^1\}_i$. Let us connect these by a compatible family of almost complex structures $\{J_i^t\}_i$, $t \in [0, 1]$. For $\alpha = 0$ or 1 , let us put $h_i^\alpha(\cdot, \cdot) = \omega_i(\cdot, J_i^\alpha \cdot)$. Then $\{h_i^\alpha\}_i$ gives a family of Riemannian metrics. Moreover each J_i^α is uniquely determined by h_i^α . Now let us consider a smooth family of Riemannian metrics:

$$h_i^t = h_i^0 + t(h_i^1 - h_i^0).$$

Then for each i , there exists a unique and smooth family of compatible almost complex structures J_i^t , $t \in [0, 1]$ satisfying $h_i^t(\cdot, \cdot) = \omega_i(\cdot, J_i^t \cdot)$.

Let us choose a complete almost Kaehler chart at $p \in M_i \subset M_{i+1}$:

$$\omega_i = \sum_{j \leq i} dp_j \wedge dq_j, \quad \omega_{i+1} = \sum_{j \leq i+1} dp_j \wedge dq_j \quad \text{at } p$$

and denote the local projections by $\pi'_i : D^{2d_{i+1}}(\epsilon) \mapsto D^{2d_i}(\epsilon)$. Let us check the compatibility condition $J_{i+1}^t \circ I(i)_* = I(i)_* \circ J_i^t$ at p and for each t . Let us take $v_i \in T_p M_i$. Then:

$$\begin{aligned} \omega_{i+1}(\cdot, J_{i+1}^t v_i) &= h_{i+1}^0(\cdot, v_i) + t(h_{i+1}^1(\cdot, v_i) - h_{i+1}^0(\cdot, v_i)) \\ &= \omega_{i+1}(\cdot, J_{i+1}^0 v_i) + t\{\omega_{i+1}(\cdot, J_{i+1}^1 v_i) - \omega_{i+1}(\cdot, J_{i+1}^0 v_i)\} \\ &= \omega_{i+1}(\cdot, J_i^0 v_i) + t\{\omega_{i+1}(\cdot, J_i^1 v_i) - \omega_{i+1}(\cdot, J_i^0 v_i)\} \\ &= \omega_{i+1}(\cdot, J_i^t v_i) \\ &= \omega_i(\pi'_i \cdot, J_i^0 v_i) + t\{\omega_i(\pi'_i \cdot, J_i^1 v_i) - \omega_i(\pi'_i \cdot, J_i^0 v_i)\} \\ &= \omega_i(\pi'_i \cdot, J_i^t v_i). \end{aligned}$$

The fourth equality implies the the compatibility condition.

Moreover the following equalities hold from the equality between the first and the last above:

$$\begin{aligned}\omega_i(\pi'_i J_{i+1}^t(w), J_i^t \pi'_i(v)) &= \omega_{i+1}(J_{i+1}^t(w), J_{i+1}^t \pi'_i(v)) \\ &= \omega_{i+1}(w, \pi'_i(v)) = \omega_i(\pi'_i(w), \pi'_i(v)) = \omega_i(J_i^t(\pi'_i(w)), J_i^t(\pi'_i(v))).\end{aligned}$$

This implies the equality:

$$\pi'_i J_{i+1}^t = J_i^t \pi'_i$$

and so π_i are holomorphic with respect to J^t . This completes the proof.

Remark 1.1: It is not clear whether the conclusion might still hold when we do not fix a uniformly bounded covering.

1.B.2 Embeddings of almost Kaehler sequences: Let $[(M_i, \omega_i, J_i)]$ be an almost Kaehler sequence equipped with complete local charts $\varphi(p) : \cup_{s \geq 1} D^{2s}(\epsilon) \hookrightarrow \text{im } \varphi(p) \subset M$ for all $p \in M$.

Let us say that $[(M'_i, \omega'_i, J'_i)]$ is *embeddable* into $[(M_i, \omega_i, J_i)]$, if there are positive $\epsilon > 0$, subindices $\{k(i)\}_i$ with $k(i) \geq i$ and compatible embeddings between almost Kaehler manifolds:

$$I_i : (M'_i, \omega'_i, J'_i) \hookrightarrow (M_{k(i)}, \omega_{k(i)}, J_{k(i)})$$

so that there are families of holomorphic maps:

$$\pi_i : U_i \mapsto M'_i$$

from the open ϵ tubular neighbourhoods $I_i(M'_i) \subset U_i \subset M \equiv \cup_i M_i$, which satisfy the properties:

$$\pi_i|_{M'_i} = \text{id} , \quad \pi_i(\varphi(p)(m)) = \varphi(p)(\tilde{\pi}_i(m))$$

for all $m \in U_i$, where $\tilde{\pi}_i : \cup_{s \geq 1} D^{2s}(\epsilon) \rightarrow D^{2d_i}(\epsilon)$ are the projections with $d_i = \dim M'_i$.

Example 1.2: Let us fix $p \geq 1$ and consider the canonical embeddings of the Grassmannians $Gr_{p,q} \hookrightarrow Gr_{p,q+1}$ which embed each p plane $L \subset \mathbf{C}^{p+q} \subset \mathbf{C}^{p+q+1}$. These admit the canonical and compatible Kaehler forms, and the direct limits $Gr_p \equiv \lim_{q \rightarrow \infty} Gr_{p,q}$ consist of the Kaehler sequences.

Let us consider the Plücker embedding $Gr_{p,q} \hookrightarrow \mathbf{CP}^N$, where $N = N(p, q) = \binom{p+q}{p} - 1$, which associate each p plane $L \subset \mathbf{C}^{p+q}$ and its basis $\{v_1, \dots, v_p\}$ to the complex line $[v_1 \wedge \dots \wedge v_p] \in \mathbf{CP}^N$.

It is well known that these embeddings preserve the canonical Kaehler forms, and so they give the embedding of the Kaehler sequences:

$$I : [Gr_{p,q}] \hookrightarrow [\mathbf{CP}^N]$$

where $(M_i, \omega_i, J_i) = Gr_{p,i}$ and $(M'_i, \omega'_i, J'_i) = \mathbf{CP}^i$ with $k(i) = N(p, i)$.

Moreover the Schubert calculus verifies the isomorphisms:

$$I_* : H_2(Gr_{p,q} : \mathbf{Z}) \cong H_2(\mathbf{CP}^N : \mathbf{Z}) \cong \mathbf{Z}$$

between simply connected spaces.

Example 1.3: Let us consider the *Veronese maps* defined as follows. Let us introduce the lexicographic order for two indices (i_0, \dots, i_n) and (j_0, \dots, j_p) .

Let us fix $m \in \{1, 2, \dots\}$, and take \mathbf{CP}^n with the homogeneous coordinate $[z_0, \dots, z_n]$. For $N = \binom{n+m}{m} - 1$, we define the Veronese map:

$$\begin{aligned} v_m : \mathbf{CP}^n &\hookrightarrow \mathbf{CP}^N, \\ v_m([z_0, \dots, z_n]) &= \{z_0^{i_0}, \dots, z_n^{i_n} : \sum_{l=0}^n i_l = m\}. \end{aligned}$$

With $n_1 = 1$, let us define numbers inductively by $n_{i+1} = \binom{n_i + m}{m} - 1$.

Now we have two different embeddings:

$$\mathbf{CP}^{n_i} \subset_v \mathbf{CP}^{n_{i+1}}, \quad \mathbf{CP}^{n_i} \subset \mathbf{CP}^{n_{i+1}}$$

where the first is the given by the Veronese map and the second is by the canonical embedding.

Lemma 1.5 *The following diagram commutes:*

$$\begin{array}{ccc} \mathbf{CP}^{n_i} & \subset_v & \mathbf{CP}^{n_{i+1}} \\ \cap & & \cap \\ \mathbf{CP}^{n_{i+1}} & \subset_v & \mathbf{CP}^{n_{i+2}} \end{array}$$

Proof: This follows since we have used the lexicographic order for the coordinates. This completes the proof.

Corollary 1.1 *There is a canonical embeddings of \mathbf{CP}^∞ to itself:*

$$v_m : \mathbf{CP}^\infty \subset_v \mathbf{CP}^\infty$$

of degree m , so that the restrictions are given by the Veronese maps.

Remark 1.2: We have the Veronese sequence by the embeddings by the iterations of the Veronese maps:

$$\mathbf{CP}^{n_1} \subset_v \mathbf{CP}^{n_2} \subset_v \cdots \subset_v \mathbf{CP}^{n_l} \subset \cdots \subset V \equiv \cup_i \mathbf{CP}^{n_i}.$$

This is not Kaehler sequence, since the degree grows unboundedly in the total space. Study of this embeddings will require much harder analysis.

1.B.3 Symmetric almost Kaehler sequence: Let us introduce geometric classes of almost Kaehler sequences. Their symmetric properties will allow us to analyze structure of holomorphic maps.

Definition 1.4 *$[(M_i, \omega_i, J_i)]$ is a symmetric almost Kaehler sequence, if there are $\epsilon > 0$, uniformly bounded coverings $\{\varphi(p)\}$ at any $p \in M$ and some $l = l(k) > k$ for any $k \geq 0$ so that there are families of almost Kaehler submanifolds $M_k \subset W_i \subset M_i$ with $W_l = M_l$ for all $i \geq l$ and isomorphisms:*

$$P_i : \{(M, M_k), \omega, J\} \cong \{(M, M_k), \omega, J\}$$

which preserve M_k and transform W_i to M_l as:

$$P_i : (W_i, \omega_i|_{W_i}, J_i|_{W_i}) \cong (M_l, \omega_l, J_l)$$

such that at any $p \in M_k$:

$$D : TM_k \oplus_{i \geq l} N_{i,k} \cong TM|_{M_k}$$

give the uniform isomorphisms over M_k with respect to the complete local charts, where:

$$\begin{aligned} N_{i,k} &= (P_i^{-1})_* [(Ker (\pi_k)_* \cap TM_l)|_{M_k}] \\ D_p &= id \oplus (P_l)_* \oplus (P_{l+1})_* \oplus \dots \end{aligned}$$

We say that the family of maps $\{(P_i, \pi_k)\}_{i,k}$ give symmetry of the almost Kaehler sequence with respect to (M_k, M_l) .

If all these properties hold by use of complex structure, then we say that it is a symmetric Kaehler sequence.

Suppose $[(M_i, \omega_i, J_i)]$ is symmetric. It is *isotropic*, if there are families of parametrized isomorphisms for each $0 \leq t \leq 1$:

$$P_i^t : \{(M, M_k), \omega, J\} \cong \{(M, M_k), \omega, J\}$$

with:

$$P_i^0 \equiv \text{id}, \quad P_i^1 = P_i.$$

Examples 1.4: (1) Let (X, ω, J) and (Y, τ, I) be two almost Kaehler manifolds, and choose a base point $y_0 \in Y$. Let us consider the products:

$$(X \times Y_1 \times Y_2 \times \dots, \omega + \tau_1 + \tau_2 + \dots, J \oplus I_1 \oplus I_2 \oplus \dots)$$

where all (Y_i, τ_i, I_i) are the same (Y, τ, I) , and we embed $X \times Y_1 \subset X \times Y_1 \times Y_2$ by identifying $X \times Y = X \times Y \times \{y_0\}$ and similar for others.

The infinite product sequence admits symmetric structure by choosing:

$$M_k = X \times Y_1 \times \dots \times Y_k, \quad W_i = M_k \times \{y_0\} \times \dots \times \{y_0\} \times Y_i.$$

P_i are given by the obvious exchange of the coordinates.

(2) Let M be a complex manifold, and take any holomorphic curve $u : \mathbf{CP}^1 \mapsto M$. Then the holomorphic vector bundle $u^*(TM) \mapsto \mathbf{CP}^1$ splits as the direct sum of holomorphic line bundles. This fact can be regarded as ‘infinitesimal symmetric property’ (see [OSS]).

(3) Let us consider the projective spaces with the Fubini Study forms $[(\mathbf{CP}^i, \omega_i)] = \mathbf{CP}^1 \subset \mathbf{CP}^2 \subset \dots \subset \mathbf{CP}^n \subset \dots \mathbf{CP}^\infty$. This is an isotropic symmetric Kaehler sequence, and we denote it by \mathbf{CP}^∞ . There are standard charts $\mathbf{C}^i \subset \mathbf{CP}^i$ and ω_i can be expressed as:

$$\omega_i|_{\mathbf{C}^i} = \frac{\sqrt{-1}}{2} \left[\frac{\sum_l dw_l \wedge d\bar{w}_l}{(1 + w\bar{w})} - \frac{(\sum_l \bar{w}_l dw_l) \wedge (\sum_l w_l d\bar{w}_l)}{(1 + w\bar{w})^2} \right]$$

where $w = (w_1, \dots, w_i)$ are the coordinates on \mathbf{C}^i . Then $[\omega_i \equiv \omega|D^{2i}]$ is completely of bounded geometry, where $D^{2i} \subset \mathbf{C}^i$ are the unit balls. In order to obtain another charts at any $p \in \mathbf{CP}^i$, one may use any constant unitary matrix $U \in \text{Mat}_{i+1}(\mathbf{C})$ with $U([1, 0, \dots, 0]) = p \in \mathbf{CP}^i$.

Let $U_\epsilon(\mathbf{CP}^k) \subset \cup_i \mathbf{CP}^i$ be ϵ tublar neighbourhood. Then there are natural projections, $\pi_k : U_\epsilon(\mathbf{CP}^k) \mapsto \mathbf{CP}^k$:

$$\pi_k([z_0, \dots, z_k, z_{k+1}, \dots]) = [z_0, \dots, z_k, 0, \dots].$$

Let us put $M_k = \mathbf{CP}^k$ and W_i by:

$$W_i = \{[z_0 : \dots : z_k : 0 : \dots : 0 : z_i : 0 : 0 : \dots] \in \mathbf{CP}^\infty\}$$

for all $i \geq l = k + 1$ with $M_l = \mathbf{CP}^{k+1}$. $P_i : W_i \cong \mathbf{CP}^{k+1}$ are given just by exchange of the coordinates:

$$[z_0 : \dots : z_k : 0 : \dots : 0 : z_i : 0 : \dots] \rightarrow [z_0 : \dots : z_k : z_i : 0 : \dots].$$

This is isotropic, by putting:

$$\begin{aligned} P_i^t([z_0 : \dots : z_k : \dots]) &= [z_0 : \dots : z_k : \cos \frac{\pi t}{2} z_{k+1} + \sin \frac{\pi t}{2} z_i : \\ & \quad z_{k+2} : \dots : z_{i-1} : -\sin \frac{\pi t}{2} z_{k+1} + \cos \frac{\pi t}{2} z_i : z_{i+1} : \dots]. \end{aligned}$$

(4) There are many variants. For example one can change \mathbf{C} by \mathbf{H} . For others, let us consider the Grassmannians:

$$Gr_{r,n}(\mathbf{C}) = \{H \subset \mathbf{C}^{r+n} ; H : r \text{ dimensional } \mathbf{C} \text{ vector subspaces} \}.$$

One can canonically embed as $H \subset \mathbf{C}^{r+n+1}$, and by taking the direct limit, one obtains the Kaehler sequence $Gr_r(\mathbf{C}) = \lim_{n \rightarrow \infty} Gr_{r,n}(\mathbf{C})$ equipped with the standard Kaehler structure.

This space also admits isotropic and symmetric structure. Let us put:

$$\mathbf{C}^{k,i} = \{(z_1, \dots, z_k, 0, \dots, 0, z_{k+i}) : z_j \in \mathbf{C}\} \subset \mathbf{C}^{k+i}$$

and choose $M_k = Gr_{r,k}$ and $W_i \equiv W_{k,i}^r$ are consisted by all elements of the form:

$$W_{k,i}^r = \{H \subset \mathbf{C}^{k+r,i} ; H : r \text{ dimensional } \mathbf{C} \text{ vector subspaces} \}.$$

The required isomorphisms and isotropies can be obtained by the same way as the above P_i and P_i^t .

Lemma 1.6 *Let $[(M_i, \omega_i, J_i)]$ be a symmetric almost Kaehler sequence, and choose any pair (M_k, M_l) as above.*

Then there is a bundle $N_{k,l} \rightarrow M_k$ with the uniform isomorphisms:

$$TM|_{M_k} \cong TM_k \oplus (N_{k,l} \otimes \mathbf{R}^\infty)$$

with respect to the complete local charts over M .

Proof: Let us put:

$$N_{k,l} = (\text{Ker } (\pi_k)_* \cap TM_l)|_{M_k}.$$

There are holomorphic isomorphisms $TM_l|_{M_k} \cong TM_k \oplus N_{k,l}$. Then the conclusion follows by use of the isomorphisms of the tangent bundles:

$$TM_k \oplus N_{k,l} \cong TW_i|_{M_k}, \quad (v, w) \rightarrow (v, (P_i^{-1})_*(w)).$$

This completes the proof.

1.B.4 Quasi transitivity: Let $[(M_i, \omega_i, J_i)]$ be an almost Kaehler sequence. We say $[(M_i, \omega_i, J_i)]$ is *quasi transitive*, if for any $N > 0$, there is $k = k(N)$ so that for any mutually different points $p_0, \dots, p_{N-1} \in M \equiv \cup_i M_i$, there are automorphisms of the almost Kaehler sequence $A : (M, \omega, J) \cong (M, \omega, J)$ such that:

$$A(p_i) \in M_k$$

hold for all $0 \leq i \leq N - 1$.

Lemma 1.7 *The infinite projective space $[(\mathbf{CP}^i, \omega_i, J_i)]$ is quasi transitive.*

Proof: Let us construct automorphisms $A^i : (M, M_{l_i}) \cong (M, M_{l_i})$ inductively so that they satisfy the followings:

$$A^i(p_i) \in M_{l_i}, \quad A^i|_{M_{l_j}} = id$$

for all $j < i$. We put $A \equiv A^{N-1} \circ A^{N-2} \circ \dots \circ A^0$ and $k = l_{N-1}$.

Let us choose any $p_0 = [z_0, z_1, \dots] \in \mathbf{CP}^L \subset \mathbf{CP}^\infty$. Firstly let us move p_0 to $[1, 0, 0, \dots]$ by a unitary automorphism $A^0 \in U(L+1) \subset \text{Aut } \mathbf{CP}^\infty$.

Let us consider $u_1 = A^0(p_1) \in \mathbf{CP}^\infty$. We put $A^1 = \text{id}$, if $u_1 \in \mathbf{CP}^1$. Suppose $u_1 = [u_1^0, u_1^1, \dots] \notin \mathbf{CP}^1$. Then (u_1^1, u_1^2, \dots) is non zero and so defines an element in \mathbf{CP}^∞ . Let us choose another unitary automorphism T_1 with $T_1([u_1^1, u_1^2, \dots]) = [1, 0, \dots]$. Then we put $A^1 = \text{diag}(1, T_1)$.

Let us consider $u_2 = A^1 \circ A^0(p_2) \in \mathbf{CP}^\infty$. We put $A^2 = \text{id}$, if $u_2 \in \mathbf{CP}^2$. Suppose $u_2 = [u_2^0, u_2^1, \dots] \notin \mathbf{CP}^2$. Then (u_2^2, u_2^3, \dots) defines an element in \mathbf{CP}^∞ . By another unitary automorphism T_2 with $T_2([u_2^2, u_2^3, \dots]) = [1, 0, \dots]$ Then we put $A^2 = \text{diag}(1, 1, T_2)$.

By the same way one can inductively construct A^3, \dots, A^{N-1} .

This completes the proof.

Remark 1.3: (1) A similar argument can be used to verify that the infinite Grassmannians $Gr_N(\mathbf{C}) = \lim_{L \rightarrow \infty} Gr_{N,L}$ also satisfy quasi transitivity.

(2) For our later applications, it is enough to assume *weakly quasi transitivity*, in the sense that the above $k = k(N, L)$ can also depend on $L = \max_{i,j} d(p_i, p_j)$.

Notice that if the diameter of M are bounded, then it is quasi transitive whenever weakly quasi transitive.

1.B.5 Minimality: Let $[(M_i, \omega_i, J_i)]$ be an almost Kaehler sequence. Let us introduce its invariant ([HV]):

$$m([(M_i, \omega_i, J_i)]) = \inf_u \{ \langle \omega, u \rangle ; \\ u : S^2 \mapsto M \equiv \cup_i M_i : \text{non constant holomorphic curve} \}.$$

By restriction to the symplectic sequence, one obtains the invariant:

$$m([(M_i, \omega_i)]) = \inf_\alpha \{ \langle \omega, \alpha \rangle > 0 : \alpha : S^2 \mapsto \cup_i M_i \}.$$

We say $[(M_i, \omega_i, J_i)]$ is *minimal*, if both the equality and positivity hold:

$$m([(M_i, \omega_i)]) = m([(M_i, \omega_i, J_i)]) > 0.$$

The minimal homotopy class $\alpha \in \text{Homot}\{S^2 \mapsto M\}$ is given by the equality $\int_\alpha \omega = m([(M_i, \omega_i, J_i)])$, which plays an important role in section 3.

Examples 1.5: (1) Notice that if $[(M_i, \omega_i, J_i)]$ is of π_2 rank 1, $\pi_2(\cup_i M_i)/\text{Tor} \cong \mathbf{Z}$, then minimality is equivalent to existence of non constant holomorphic curves representing $1 \in \pi_2/\text{Tor}$.

The Fubini Study form on CP^n with the standard complex structure gives π_2 rank one minimal data (ω, J) with $m = \pi$.

(2) Let $(\mathbf{CP}^1, \omega, J)$ be the standard curve and $[(M_i, \omega_i, J_i)]$ be minimal. Then the product $[(M_i \times \mathbf{CP}^1, \omega_i + \omega, J_i \oplus J)]$ is also minimal.

(3) Suppose $[(M_i, \omega_i, J_i)]$ is algebraic with each $\omega_i \in H^2(M_i : \mathbf{Z})$. Then it is minimal, if any generating elements in $H_2(M : \mathbf{Z})$ can be represented by some holomorphic curves. In particular it is the case when it is simply connected, algebraic, and any generating elements in $\pi_2(M)$ can be represented by some holomorphic curves.

1.C Transition functions: Let $[(M_i, \omega_i, J_i)]$ be an almost Kaehler sequence, and choose a uniformly bounded covering $\{(p, \varphi(p))\}$.

For any $p, p' \in M = \cup_i M_i$, let us put:

$$\Phi(p, p') \equiv \varphi(p')^{-1} \circ \varphi(p) : B(p, p') \cong B(p', p)$$

where $B(p, p') \equiv \varphi(p)^{-1}(\text{im } \varphi(p) \cap \text{im } \varphi(p')) \subset \cup_i D^{2i}(\epsilon)$.

Lemma 1.8 *Let $[(M_i, \omega_i, J_i)]$ be an almost Kaehler sequence. Then $\Phi(p, p')$ give the complete isomorphisms:*

$$\sup_{p, p'} \|\nabla^l \Phi(p, p')\|^{C^0} \leq c_l$$

for constants c_l independently of p, p' for all $l \geq 0$.

Proof: See def 1.1 and below it for the terminologies.

This can be seen by use of the exponential mappings ([Kl] p74). For convenience we check the uniform estimates for $l \leq 2$, which will be used to construct diffeomorphisms in lemma 5.3 below.

The estimates on the first derivative come from uniformity of compatible Riemannian metrics.

Let us put $g(p) = \varphi(p)^*(g)$ on $\cup_i D^{2i}(\epsilon')$. Thus $\Phi(p, p')^*(g(p')) = g(p)$ holds. Since both of $g(p)$ and $g(p')$ are uniformly bounded, the conclusion holds for $l = 1$. Let us put $\Phi = \Phi(p, p')$.

Suppose $\nabla\Phi$ could be unbounded at m , and take a C^1 curve $\gamma : [-1, 1] \rightarrow B(p, p')$ with $\gamma(0) = m \in B(p, p')$.

Let $X_0 \in T_m M$ be a unit tangent vector with $|X_0| = 1$, and extend X_0 as the vector fields X along γ by constant. Then we put $W_0 = \frac{d\Phi_*(X_s)}{ds}|_{s=0} \in T_{\Phi(m)} M$ and extend W_0 to W along $\Phi(\gamma)$ similarly. We also have another vector field $Z = \Phi^{-1}(W)$ along γ . Notice $\frac{dX}{ds} = \frac{dW}{ds} = 0$ and $\frac{dZ}{ds}|_{s=0} = X_0$.

Let us consider the equalities:

$$g(p)(X, Z) = \Phi^*(g(p'))(X, Z) = g(p')(\Phi_*(X), W).$$

By differentiating both sides at $s = 0$, one has the equality:

$$g(p)'(X_0, \Phi^{-1}(W_0)) + g(p)(X_0, X_0) = g(p')'(\Phi_*(X_0), W_0) + g(p')(W_0, W_0)$$

where $g(p)' = dg(p)(\gamma(s))/ds|_{s=0}$. So there is a constant C with the point wise estimates:

$$\|\nabla\Phi_*(X_0)\|^2 \leq C\{\|\nabla\Phi_*(X_0)\| + 1\}$$

which contradicts to the assumption.

2 Moduli spaces of holomorphic curves

In this section we study theory of holomorphic curves into almost Kaehler sequences. In particular we develop the analytic tools to construct finite dimensional moduli spaces over sequences which satisfy some symmetric properties.

2.A Finite dimensional preliminaries: We recall basics of moduli theory of holomorphic curves into finite dimensional symplectic manifolds. Most of the contents in 2.A are already in [HV], which are preliminaries for 2.B where we formulate Sobolev spaces over the infinite dimensional spaces $M = \cup_i M_i$.

\mathbf{CP}^1 has particular points $0, \infty \in \mathbf{CP}^1$, and let $0 \in D(1) \subset S^2 = \mathbf{CP}^1$ be the hemi sphere. We choose and fix the following data:

- (1) a large $l \geq 1$,

- (2) a non trivial homotopy class $\alpha \in \pi_2(\cup_i M_i)$, and
- (3) different fixed points $p_0, p_\infty \in M_0 \subset M \equiv \cup_i M_i$.

Let $L_{l+1}^2(S^2, M_i)$ be the sets of L_{l+1}^2 maps from S^2 to M_i . (In 2.B, we will define them in detail). Let us put the spaces of Sobolev maps:

$$\mathfrak{B}_i \equiv \mathfrak{B}_i(\alpha) = \{u \in L_{l+1}^2(S^2, M_i) : [u] = \alpha, \int_{D(1)} u^*(\omega) = \frac{1}{2} \langle \omega, \alpha \rangle, \quad u(*) = p_* \in M_0, * \in \{0, \infty\}\}.$$

Let $E(J)_i, F_i \mapsto S^2 \times M_i$ be vector bundles whose fibers are respectively:

$$\begin{aligned} E(J)_i(z, m) &= \{\phi : T_z S^2 \mapsto T_m M_i : \text{anti complex linear}\}, \\ F_i(z, m) &= \{\phi : T_z S^2 \mapsto T_m M_i : \text{linear}\}. \end{aligned}$$

Then we have two stratified Hilbert bundles over \mathfrak{B}_i :

$$\begin{aligned} \mathfrak{E}_i &= L_l^2(\mathfrak{B}_i^*(E(J)_i)) = \cup_{u \in \mathfrak{B}_i} \{u\} \times L_l^2(u^*(E(J)_i)), \\ \mathfrak{F}_i &= L_l^2(\mathfrak{B}_i^*(F_i)) = \cup_{u \in \mathfrak{B}_i} \{u\} \times L_l^2(u^*(F_i)). \end{aligned}$$

On all of these Hilbert manifolds $\mathfrak{B}_i, \mathfrak{E}_i, \mathfrak{F}_i$, there exist compatible, free and continuous S^1 actions which are induced from the one on $\mathbf{C} \subset \mathbf{CP}^1$.

Remark 2.1: One may regard that $E(J) = \cup_i E(J)_i$ and $F = \cup_i F_i$ are stratified vector bundles over $S^2 \times M$, $M = \cup_i M_i$, and so we have stratified Hilbert bundles $\mathfrak{E} = \cup_i \mathfrak{E}_i$ and $\mathfrak{F} = \cup_i \mathfrak{F}_i$ over $\mathfrak{B} \equiv \cup_i \mathfrak{B}_i$.

The non linear *Cauchy-Riemann operators* and their tangent maps are defined respectively as sections:

$$\begin{aligned} \bar{\partial}_J &\in C^\infty(\mathfrak{E}_i \mapsto \mathfrak{B}_i), \quad \bar{\partial}_J(u) = Tu + J \circ Tu \circ i, \\ T &\in C^\infty(\mathfrak{F}_i \mapsto \mathfrak{B}_i), \quad T(u) = Tu, \quad u \in \mathfrak{B}_i \end{aligned}$$

where i is the complex conjugation on $S^2 = \mathbf{CP}^1$. If u satisfies $\bar{\partial}_J(u) = 0$, then we say that u is a *holomorphic curve* or *J-curve*.

Now let us define the moduli space of holomorphic curves by:

$$\mathfrak{M}(\alpha, M_i, J_i) = \{u \in C^\infty(S^2, M_i) \cap \mathfrak{B}_i(\alpha) : \bar{\partial}_J(u) = 0\}.$$

There is an induced S^1 free action on \mathfrak{M}_i .

We say that J is *regular*, if for any $u \in \mathfrak{M}_i$, the linealizations:

$$D\bar{\partial}_J(u) : T_u \mathfrak{B}_i \mapsto (\mathfrak{E}_i)_u$$

are onto for all i .

The following estimates follow from the inverse function theorem and the Riemann Roch theorem:

Proposition 2.1 *Let $[(M_i, \omega_i, J_i)]$ be a regular almost Kaehler sequence. Then the moduli spaces are S^1 free manifolds with dimension:*

$$\dim \mathfrak{M}(\alpha, M_i, J_i) = 2 < c_1(T^{1,0} M_i), [u] > -2 \dim M_i - 1.$$

If moreover it is minimal, then each $\mathfrak{M}(\alpha, M_i, J_i)$ is compact.

Later on we will omit to denote α explicitly.

Definition 2.1 *The moduli spaces of holomorphic curves for almost Kaehler sequences are given by:*

$$\mathfrak{M}([(M_i, \omega_i, J_i)]) = \cup_{i \geq 1} \mathfrak{M}(\alpha, M_i, J_i).$$

Lemma 2.1 *Let $[(M_i, \omega_i, J_i)]$ be a regular almost Kaehler sequence. Then $\mathfrak{M}([(M_i, \omega_i, J_i)])$ is a S^1 freely stratified manifold.*

Example 2.1: Let us consider the standard holomorphic embedding $\mathbf{CP}^1 \hookrightarrow \mathbf{CP}^n$ with fixed two points. Modulo S^1 action, this is the unique element in the moduli space which is regular in the minimal class.

2.B Sacks-Uhlenbeck's estimates:

Lemma 2.2 *Let $[(M_i, \omega_i, J_i)]$ be a minimal almost Kaehler sequence. For each α , there is a constant $c(\alpha) \geq 0$ so that any $u \in \mathfrak{M}([(M_i, \omega_i, J_i)])$ satisfy uniform estimates:*

$$|\nabla^\alpha u|_{C^0(S^2)} \leq c(\alpha).$$

Proof: We will only verify the uniform estimate $|\nabla u|_{C^0} \leq c$. The estimates for higher devrivatives follow from the elliptic regularity.

There is a biholomorphic isomorphism:

$$\Phi : Z = \mathbf{R} \times S^1 \cong \mathbf{CP}^1 \setminus \{0, \infty\}, \quad (r, t) \rightarrow \exp(r + 2\pi it)$$

where we equip the standard complex structure on Z . For any holomorphic curve $u \in \mathfrak{M}$, let us regard $u : \mathbf{R} \times S^1 \mapsto M$ with $u(-\infty) = p_0$ and $u(\infty) = p_\infty \in M_0$.

It follows from $\frac{\partial}{\partial s}u + J(u)\frac{\partial}{\partial t}u = 0$ that the equalities hold:

$$||du||^2 = \omega\left(\frac{\partial}{\partial s}u, J(u)\frac{\partial}{\partial s}u\right) + \omega\left(\frac{\partial}{\partial t}u, J(u)\frac{\partial}{\partial t}u\right) = 2\omega\left(\frac{\partial}{\partial s}u, \frac{\partial}{\partial t}u\right) = 2||u^*(\omega)||^2.$$

Sublemma 2.1 (SU) *There are constants C and $\epsilon > 0$ determined by $[(M_i, \omega_i, J_i)]$ so that for any holomorphic disk $u : D^2 \mapsto M = \cup_i M_i$ and $E = \int_{D^2} u^*(\omega) \leq \epsilon$, the estimate holds:*

$$\varphi(x) \leq CE, \quad \varphi = |du|^2$$

for all $x \in D^2(\frac{1}{2})$.

Proof of the lemma: Let us fix a small positive constant $\delta > 0$. Then for any $u \in \mathfrak{M}([(M_i, \omega_i, J_i)])$, we put $s(u) \equiv s_\infty(u) - s_0(u) > 0$, where:

$$\begin{aligned} s_0(u) &= \sup\{s \in \mathbf{R} : d(u((-\infty, s) \times S^1), p_0) \leq \delta\}, \\ s_\infty(u) &= \inf\{s \in \mathbf{R} : d(u((s, \infty) \times S^1), p_\infty) \leq \delta\}. \end{aligned}$$

Step 1: We claim that for $0 < \mu \leq \frac{s(u)}{3}$, there is a positive $\epsilon > 0$ determined by $[(M_i, \omega_i, J_i)]$ and μ with the estimates:

$$\int_{(-\infty, s_0(u)+\mu] \times S^1} u^*(\omega), \quad \int_{[s_\infty(u)-\mu, \infty) \times S^1} u^*(\omega) \geq \epsilon.$$

We will only verify the first estimate. The latter follows by the same argument. Notice that the translation on Z is an automorphism (which does not preserve the required condition $\int_{D(1)} u^*(\omega) = \frac{1}{2} < \omega, \alpha > \text{ on } \mathfrak{B}$).

Let us choose a translation T on Z so that $s_0(u \circ T) = 0$ holds. Notice $s(u \circ T) = s(u) \geq 3\mu$. One may assume $s_0(u) = 0$, since the equality:

$$\int_{(-\infty, s_0(u \circ T)+\mu] \times S^1} (u \circ T)^*(\omega) = \int_{(-\infty, s_0(u)+\mu] \times S^1} u^*(\omega)$$

holds. Let $D^2(b) \subset S^2$ be the disk with the radius b . Let $a > 0$ be as:

$$(-\infty, s_0(u) + \mu] \times S^1 = D^2(1+a) \setminus 0 \subset S^2$$

where we identify $(-\infty, s_0(u)] \times S^1 = D^2(1) \setminus 0$. We put $D = D^2(1)$ and $D' = D^2(1+a)$.

Let us put $B_\delta(0) \equiv \{m \in M : d(p_0, m) < \delta\} \subset M$ as δ neighbourhood of p_0 . Then $u(s, t) \in \partial B_\delta(0)$ and so the equality $d(u(s, t), u(-\infty)) = \delta$ holds at $s = s_0(u)$ and some $t \in S^1$ with $(s, t) \in \partial D$.

Suppose $\int_{D'} u^*(\omega) < \epsilon$ for sufficiently small $\epsilon = \epsilon(\mu) > 0$. Then by sublemma 2.1, the uniform estimates of the derivative:

$$|du| \leq C(\mu)\sqrt{\epsilon}$$

hold on all points of D . This is a contradiction if $\epsilon > 0$ are too small, since $u(s, t) \in \partial B_\delta(0)$ and $d(p_0, u(s, t)) = \delta$ as above.

This completes the proof of the claim.

Step 2: Let us proceed the proof of the lemma by the contradiction argument. Suppose contrary. Then there are families $\{u_i\}_i \subset \mathfrak{M}([(M_i, \omega_i, J_i)])$ and $\{p_i\}_i \subset S^2$ with $|\nabla u_i|(p_i) \rightarrow \infty$. As [HV] p611, one may assume the inequalities:

$$\begin{aligned} |\nabla u_i|(x) &\leq 2|\nabla u_i|(p_i) \text{ for all } x \text{ with } d(x, p_i) \leq \epsilon_i, \\ \epsilon_i |\nabla u_i|(p_i) &\rightarrow \infty, \quad \epsilon_i \rightarrow 0. \end{aligned}$$

Let $D_i = D_i(p_i)$ be r balls with the center p_i for some small $r > 0$. We put rescaled balls with the center p_i as $B_i = |\nabla u_i|(p_i)D_i(p_i)$ by multiplying the numbers $|\nabla u_i|(p_i)$, where one regards $B_i \subset \mathbf{C}$. By conformal invariance, one gets a family of holomorphic maps $v_i : B_i \mapsto M = \cup_i M_i$. This family satisfies uniform bounds:

$$|dv_i|(0) = 1, \quad |dv_i|(x) \leq 2 \quad \text{for } |x| \leq \epsilon_i |\nabla u_i|(p_i).$$

In particular by choosing small $1 \gg a, \epsilon' > 0$, $||dv_i|^2(x) - |dv_i|^2(0)| \leq \epsilon'$ holds for all $x \in D(a)$ by the elliptic regularity, where $D(a) \subset B_i$ is a ball with the center 0, and a is independent of i . This implies the estimates

$|dv_i|(x) \geq \sqrt{1 - \epsilon'}$. In particular the uniform estimates hold from below:

$$\int_{D(b_i)} v_i^*(\omega) \geq \int_{D(a)} v_i^*(\omega) \geq C > 0$$

for all $b_i \geq a$ with $D(b_i) \subset B_i$.

Step 3: On the other hand the uniform bounds $\int_{B_i} v_i^*(\omega) \leq m$ hold from above where m is the minimal invariant. We claim that there is some family $\{R_i \leq \epsilon_i |\nabla u_i|(p_i)\}$ with $R_i \rightarrow \infty$ such that the length δ_i of $x_i \equiv v_i(\exp(2\pi i R_i t)) : S^1 \mapsto M$ must decay $\delta_i \rightarrow 0$.

Notice that $[a, b_i] \times S^1 \subset B_i$ hold. Then there are some R_i so that the decay:

$$\int_{[R_i-1, R_i+1] \times S^1} v_i^*(\omega) \rightarrow 0$$

must hold. So the decay $\sup_{x \in R_i \times S^1} |dv_i|(x) \rightarrow 0$ hold by sublemma 2.1, which verifies the claim.

Step 4: Thus there is a family of small disks $\{d_i\}_i \subset M$ which span x_i , and $\int_{d_i} \omega \rightarrow 0$. Let $B'_i \subset B_i$ be R_i balls with the center p_i , whose boundaries are x_i . Let us put two ‘almost’ holomorphic spheres:

$$u'_i = \begin{cases} u_i \text{ on } S^2 \setminus B'_i \\ d_i \end{cases}, \quad v'_i = B'_i \cup d_i.$$

By the condition, these must satisfy:

$$\begin{aligned} \langle \omega, u'_i \rangle + \langle \omega, v'_i \rangle &\rightarrow m > 0, \\ \lim_i \langle \omega, u'_i \rangle, \quad \lim_i \langle \omega, v'_i \rangle &\geq 0 \end{aligned}$$

By minimality, one of $\langle \omega, u'_i \rangle$ or $\langle \omega, v'_i \rangle$ must be zero for all large i . By step 2 and 3, $\langle \omega, v'_i \rangle$ must be positive and equal to m . So $\langle \omega, u'_i \rangle = 0$ must hold.

First of all, suppose there is a uniform lower bound $s(u_i) \geq 3\mu_0 > 0$. There are three cases;

- (1) an infinite subset of $\{p_i\}_i$ is contained in $(-\infty, s_0(u_i)] \times S^1$ or
- (2) is contained in $[s_\infty(u_i), \infty) \times S^1$ or
- (3) in $[s_0(u_i), s_\infty(u_i)] \times S^1$.

Suppose the case (1). Then by step 1, there is a positive $\epsilon > 0$ with $\int_{[s_\infty(u_i) - \mu_0, \infty) \times S^1} (u'_i)^*(\omega) \geq \epsilon$. This implies the asymptotic bounds:

$$\lim_i \langle \omega, u'_i \rangle \geq \epsilon$$

which give a contradiction. The other cases can be considered similarly.

Step 5: Let us verify that $s(u_i) \rightarrow 0$ cannot happen. This will complete the proof of the lemma. Suppose contrary. Let us take $p = 0, q = \frac{1}{2} \in S^1$. Then since $u_i(o \times s_*(u_i)) \in B_\delta(*)$, $*$ = $0, \infty$ and $o = p, q$, and since $d(B_\delta(0), B_\delta(\infty)) > 0$ is positive, there are families $\{t_i\}$ and $\{r_i\}$, $t_i, r_i \in [s_0(u_i), s_\infty(u_i)]$, such that $|\nabla u_i|(p \times t_i), |\nabla u_i|(q \times r_i) \rightarrow \infty$. On the other hand one has a lower bound $d(p \times t_i, q \times r_i) \geq \frac{1}{2}$ in $\mathbf{R} \times S^1$. By the same arguments as step 2 and 3, one obtains two non trivial almost holomorphic spheres, which also cannot happen by minimality of the homotopy class.

This completes the proof.

2.C Hilbert completion of function spaces: Here we introduce the basic function spaces for the infinite dimensional analysis.

Let us take an element:

$$u \in \mathfrak{B}_i(\alpha) \subset \mathfrak{B}(\alpha) \equiv \cup_{i \geq 1} \mathfrak{B}_i(\alpha)$$

and let $U(u) \subset \mathfrak{B}(\alpha)$ be a small neighbourhood of u in the set of L^2_{l+1} maps from S^2 to M . Below we will describe its completion to a Hilbert manifold $\hat{U}(u)$.

Let us check the Sobolev embeddings for maps into Hilbert space.

Lemma 2.3 *There are constants c_l with the uniform estimates:*

$$\|u\|_{C^{l-1}(S^2)} \leq c_l \|u\|_{L^2_{l+1}(S^2)}.$$

Proof: By uniformity of complete local charts, it is enough to verify the uniform estimates:

$$\|u\|_{C^{l-1}_c} \leq c_l \|u\|_{L^2_{l+1}}$$

for $u \in C_c(U; H)$ with open subset $U \subset \mathbf{R}^2$.

The Sobolev estimate $\|\tilde{u}\|_{C^{l-1}_c(U)} \leq c_l \|\tilde{u}\|_{L^2_{l+1}(U)}$ hold for $\tilde{u} \in C_c(U)$. Let H be the closure of \mathbf{R}^∞ with the standard norm, and express $u =$

$(\tilde{u}_0, \tilde{u}_1, \dots) \in C_c^{l-1}(U; H)$. Then we have the estimates:

$$\begin{aligned} \sum_{k=0}^{l-1} |\nabla^k u|^2(m) &= \sum_{k=0}^{l-1} \sum_{j \geq 0} |\nabla^k \tilde{u}_j|^2(m) \\ &\leq c_l \sum_{j \geq 0} \|\tilde{u}_j\|^2 L_{l+1}^2(U) = c_l \|u\|^2 L_{l+1}^2(U) \end{aligned}$$

for any $m \in U$. By taking sup of the values of the left hand side, we obtain the desired estimates. This completes the proof.

Remark 2.2: (1) Later we will find a reason why to use such completion of function spaces, rather than stratified spaces in some step by step ways, where we will use some automorphisms on almost Kaehler sequences which do not preserve stratifications.

(2) All functional spaces as Hilbert manifolds admit the free and continuous S^1 actions. In precise there is a constant $C > 0$ with the inequalities:

$$C^{-1} \|u\| \leq \sup_{t \in S^1} \|tu\| \leq C \|u\|$$

for all elements u in such Hilbert spaces.

Let $\varphi(p) : D \equiv \cup_i D^{2i}(\epsilon) \hookrightarrow \cup_i M_i$ be a complete almost Kaehler chart at p . Sometimes we will identify D and $D(p)$ where:

$$D \equiv \cup_i D^{2i}(\epsilon) \subset \mathbf{R}^\infty \subset H, \quad D(p) \equiv \varphi(p)(\cup_i D^{2i}(\epsilon)) \subset M.$$

By definition D is equipped with the induced metric which is uniformly equivalent to the standard one on H . Let us take the following data:

- (1) finite set of points $s_0, \dots, s_k \in S^2$,
- (2) an open cover U_0, \dots, U_k with $s_i \in U_i \subset S^2$, and
- (3) a partition of unity f_0, \dots, f_k over S^2 .

For $u \in \mathfrak{B}(\alpha)$, one can choose large k so that each image $u(U_j)$ is contained in a complete almost Kaehler chart at $\varphi(p_j)$ with $p_j = u(s_j)$. Then one can express its restrictions as $u|_{U_j} : (U_j, s_j) \mapsto (D(p_j), p_j)$. Identifying $D(p_j)$ with D as above, one may regard these maps as:

$$u|_{U_j} : (U_j, s_j) \mapsto (D, 0) \subset (H, 0).$$

Let us introduce precisely the Hilbert norm on the set of sections of $u^*(E(J))$ as follows; let us take any $\varphi \in \Gamma(u^*(E(J)))$. Then one may express the

restriction as:

$$\varphi|_{U_j} : TU_j \mapsto TD = D \times \mathbf{R}^\infty$$

which is anti linear with respect to $(i, J_{u(m)})$ at $(m, u(m))$. Then one can define:

$$||\varphi||^2 L_l^2 \equiv \sum_{0 \leq j \leq k} \sum_{0 \leq a \leq l} \int_{U_j} |\nabla^a(f_j \varphi|_{U_j})|^2(m) dm.$$

By taking completion with respect to the above norm, one obtains the Hilbert bundles:

$$\hat{\mathfrak{E}} = \cup_{u \in \mathfrak{B}} \hat{\mathfrak{E}}_u \equiv \cup_{u \in \mathfrak{B}} \{u\} \times L_l^2(u^*(E(J))).$$

One can also make completion of the functional spaces \mathfrak{F} and obtain $\hat{\mathfrak{F}}$.

Let $u \in \mathfrak{B} = \cup_i \mathfrak{B}_i$, and $U(u) \subset \mathfrak{B}$ be a small open subset. Let us complete $U(u)$ so that one obtains a Hilbert manifold $\hat{U}(u)$ as below. Let us write $u|_{U_j} : (U_j, s_j) \mapsto (D, 0)$ for any $u \in \mathfrak{B}_i$. Then locally any element $v \in U(u)$ can be expressed as $v|_{U_l} : U_l \mapsto \mathbf{R}^\infty \subset H$. Then we introduce Sobolev norms on $U(u)$ by:

$$||v||^2 L_{l+1}^2 = \sum_{0 \leq j \leq k} \sum_{0 \leq a \leq l+1} \int_{U_j} |\nabla^a(f_j v)|^2(m) dm \quad (*)$$

By completion, one obtains the Hilbert manifolds:

$$\hat{U}(u) \quad \text{over} \quad \cup_i M_i \quad (u \in \mathfrak{B} = \cup_{i \geq 1} \mathfrak{B}_i)$$

on neighbourhoods of u , where the local Hilbert-structures are obtained passing through the exponential map.

Notice that if u is holomorphic, then k above can be chosen uniformly by lemma 2.2.

Let us introduce:

$$\hat{\mathfrak{M}}([(M_i, \omega_i, J_i)]) = \cup_{u \in \mathfrak{M}([(M_i, \omega_i, J_i)])} \{v \in \hat{U}(u) : \bar{\partial}_J(v) = 0\}.$$

Apriori this space is bigger than the moduli space $\mathfrak{M}([(M_i, \omega_i, J_i)])$. Later we study on their coincidence each other.

2.C.2 Functional framework: Let H be a Hilbert space, and $L \subset H$ be a closed subspace.

Lemma 2.4 *Let $F : H \rightarrow H$ be a bounded operator with closed range, whose kernel consists of finite dimensional subspace. Then $F(L) \subset H$ is also closed. In particular if F is injective, then $F(L)$ is closed.*

Proof: If kernel $F = 0$, then $F : H \cong F(H)$ gives an isomorphism. In particular $F(L)$ is closed.

Suppose $\ker(F) = K \subset H$ is of finite dimension. Then F induces an isomorphism $F : H/K \cong F(H)$, where we equip with the metric on H/K by use of orthogonal decomposition $H = K^\perp \oplus K$. Then it is enough to see that the image of the projection $pr(L) \subset H/K$ is still closed.

One may assume that $L \cap K = 0$ by replacing L by $(L \cap K)^\perp$ in L , when it has positive dimension.

Suppose a sequence $\{\bar{v}_i\}_i \subset pr(L)$ converge to some element $\bar{v} \in H/K$. By the assumption, the representatives $v_i \in L$ of \bar{v}_i are unique. Let us represent $v_i = v_i^1 + v_i^2 \in L$ with respect to the decomposition $H = K^\perp \oplus K$.

We claim that $\|v_i\|$ are uniformly bounded. Suppose contrary and assume $\|v_i\| \rightarrow \infty$. Then by normalizing as $w_i = \|v_i\|^{-1}v_i = w_i^1 + w_i^2$, both $\|w_i^1\| \rightarrow 0$ and $\|w_i^2\| \rightarrow 1$ hold. Since K is finite dimensional and L is closed, a subsequence w_i converges to some element $w \in L \cap K$ with $\|w\| = 1$. This contradicts to our assumption, which verifies the claim.

Now since $\{v_i^2\}_i \subset K$ is a bounded sequence, a subsequence converges to some element $v^2 \in K$. Since v_i^1 converges to v , it follows from these that a subsequence of $\{v_i\}_i$ converges to $v + v^2 \in L$. This implies $\bar{v} \in pr(L)$.

This completes the proof.

Remark 2.3: The assumption of finite dimensionality is necessary. Let H be a separable infinite dimensional Hilbert space, and choose an orthonormal basis $\{v_i\}_i$. Let $0 < a_i \rightarrow 0$ be a decreasing family of numbers.

Let us consider the surjective bounded map:

$$F = \text{id} \oplus 0 : H \oplus H \rightarrow H$$

and the closed subspaces L spanned by the basis:

$$L = \text{span} \{w_i = (a_i v_i, v_i) : i = 0, 1, 2, \dots\} \subset H \oplus H.$$

We claim that the image of the restriction $F|L$ is not closed. Suppose contrary. Then since $F|L$ is injective, the restriction must be an isomorphism by the open mapping theorem. So there must exist some $C > 0$ with the uniform estimates:

$$|a_i v_i| = |F(w_i)| \geq C|w_i| = C\sqrt{a_i^2 + 1}.$$

But the left hand side converge to 0, which contradict to the right hand side.

The following abstract property is the key to our Fredholm theory we develop later:

Corollary 2.1 *Suppose the above situation, and choose another Hilbert space W . Then the image of the Hilbert space tensor product $L \otimes W$ over the induced operator $F \otimes 1 : H \otimes W \rightarrow H \otimes W$, still has closed range.*

In particular if F is an isomorphism, then $F \otimes 1$ is also the same.

Proof: Let us put $E = L \cap \text{Ker}(F)$, and decompose $L \cong L' \oplus E$. Then $F(L) = F(L')$ holds. Since the restriction $F|L'$ is injective, it gives the isomorphism onto $F(L)$ by the open mapping theorem.

Since the restriction $F \otimes 1|L' \otimes W$ gives the isomorphism onto $F \otimes 1(L' \otimes W) = F \otimes 1(L \otimes W)$, the conclusion follows. This completes the proof.

2.D Geometric conditions: We study functional analytic properties of the Cauchy-Riemann operators over almost Kaehler sequences which satisfy the geometric conditions we have introduced in 1.B.

Let us say that a subset $S \subset \mathfrak{B}([(M_i, \omega_i, J_i)]) = \cup_i \mathfrak{B}(M_i, \omega_i, J_i)$ is *bounded*, if there is some i_0 so that the images of u are contained in M_{i_0} for any elements $u \in S$.

Our aim in 2.D is to verify the following:

Theorem 2.1 *Let $[(M_i, \omega_i, J_i)]$ be a symmetric Kaehler sequence.*

(1) Suppose it is regular and $\dim \cup_i \text{Ker } D_u \bar{\partial}_i = N$ is finite, then it is in fact strongly regular of index N .

In particluar $\mathfrak{M}([(M_i, \omega_i, J_i)])$ is a regular N dimensional S^1 free manifold.

(2) Assume moreover it is isotropic, and each connected component of $\mathfrak{M}([(M_i, \omega_i, J_i)])$ is bounded. Then the equality holds:

$$\mathfrak{M}([(M_i, \omega_i, J_i)]) = \mathfrak{M}(M_0, \omega_0, J_0).$$

In particular if it is minimal, then $\mathfrak{M}([(M_i, \omega_i, J_i)])$ is compact.

The second condition in (2) is satisfied when $N = 1$.

The former follows from combination of lemma 2.7 and proposition 2.2 below. The latter is verified in lemma 2.8.

2.D.1 Strong regularity over symmetric Kaehler sequences: Let $[(M_i, \omega_i, J_i)]$ be a symmetric almost Kaehler sequence, and choose symmetric data $\{(\pi_k, P_i)\}_{i,k}$ with respect to $(M_k, M_{l'})$ for some $l' = l(k)$.

For any $u \in \mathfrak{B}_k \subset \mathfrak{B}_{l'} \subset \mathfrak{B}$, let $\hat{U}(u) \subset \mathfrak{B}$ be as in 2.C. Let us take small neighbourhoods $U(u)_{l'} \subset \mathfrak{B}_{l'} \cap \hat{U}(u)$.

There are extended projections:

$$\bar{\pi}_k : \hat{U}(u) \rightarrow U(u)_k$$

with $\bar{\pi}_k|_{U(u)_k} = \text{id}$. Then the isomorphism:

$$T_u U(u)_{l'} \cong T_u \mathfrak{B}_k \oplus V_u(k, l')$$

hold, where:

$$V_u(k, l') = \text{Ker } (\bar{\pi}_k)_* \cap T_u U(u)_{l'}.$$

Lemma 2.5 *The complete isomorphisms hold:*

$$T_u \hat{U}(u) \cong T_u \mathfrak{B}_k \oplus V_u(k, l') \otimes H$$

where H is a separable Hilbert space.

Proof: This follows from lemma 1.6. This completes the proof.

The Cauchy-Riemann operator $\bar{\partial}_J$ and the tangent map T give smooth sections respectively:

$$\bar{\partial}_J : \hat{U}(u) \mapsto \hat{\mathfrak{E}}|_{\hat{U}(u)}, \quad T : \hat{U}(u) \mapsto \hat{\mathfrak{F}}|_{\hat{U}(u)}.$$

Definition 2.2 Let $[(M_i, \omega_i, J_i)]$ be a regular almost Kaehler sequence. It is strongly regular, if the differential:

$$D\bar{\partial}_u : T_u\hat{U}(u) \mapsto \hat{\mathfrak{E}}_u|\hat{U}(u)$$

is surjective for any $u \in \mathfrak{M}([(M_i, \omega_i, J_i)])$.

Lemma 2.6 Let $[(M_i, \omega_i, J_i)]$ be a symmetric Kaehler sequence.

Then $D\bar{\partial}_J : T_u\hat{U}(u) \mapsto \hat{\mathfrak{E}}_u$ has closed range.

Proof: Firstly we verify that $DT : T_u\hat{U}(u) \mapsto \hat{\mathfrak{F}}_u$ has closed range, and then we verify the conclusion.

Step 1: Let us take some k so that $u \in \mathfrak{M}(M_k, \omega_k, J_k)$ with $u : S^2 \rightarrow M_k$. Let $P_i : \{(M, M_k), \omega, J\} \cong \{(M, M_k), \omega, J\}$ be the locally homogeneous data for all $i \geq l' = l(k)$.

Let us consider the bundles over $S^2 \times M_k$:

$$N_{k,l'}(m, z) = \{\phi : T_z S^2 \rightarrow \text{Ker } (\pi_k)_* \cap T_m M_{l'} : \text{linear}\} \subset F_l(z, m)$$

and put the Hilbert subbundles over \mathfrak{B}_k :

$$\mathfrak{F}_{k,l'} = L_l^2(\mathfrak{B}_k^*(N_{k,l'})) = \cup_{u \in \mathfrak{B}_k} \{u\} \times L_l^2(u^*(N_{k,l'})) \subset \mathfrak{F}_{l'}|\mathfrak{B}_k.$$

There are bundle decompositions $\mathfrak{F}_{l'}|\mathfrak{B}_k \cong \mathfrak{F}_k \oplus \mathfrak{F}_{k,l'}$ over \mathfrak{B}_k given by:

$$\bar{\phi} \rightarrow ((\pi_k)_*(\bar{\phi}), \bar{\phi} - (\pi_k)_*(\bar{\phi})).$$

It follows from symmetric property that the bundle decompositions:

$$\hat{\mathfrak{F}}|\mathfrak{B}_k \cong \mathfrak{F}_k \oplus \mathfrak{F}_{k,l'} \otimes H$$

hold as lemma 2.5. Let:

$$DT_{l'} : T_u\mathfrak{B}_{l'} = T_u\mathfrak{B}_k \oplus V_u(k, l') \rightarrow \mathfrak{F}_{l'}|_u = \mathfrak{F}_k|_u \oplus \mathfrak{F}_{k,l'}|_u$$

be the tangent map. Clearly this is diagonal $DT_{l'} = DT_k \oplus DT_{k,l'}$ with respect to these decompositions. Then the total tangent map:

$$DT : T_u\hat{U}(u) \cong T_u\mathfrak{B}_k \oplus V_u(k, l') \otimes H \rightarrow \hat{\mathfrak{F}}|_u \cong \mathfrak{F}_k|_u \oplus \mathfrak{F}_{k,l'}|_u \otimes H$$

is also diagonal:

$$DT = DT_k \oplus (DT_{k,l'} \otimes \text{id}).$$

Now $DT_l : T_u \mathfrak{B}_l \rightarrow \mathfrak{F}_l|_u$ has closed range with finite dimensional kernel, which follow from the well known analysis of holomorphic curves into finite dimensional symplectic manifolds (see [HV]). Since $V_u(k, l) \subset T_u \mathfrak{B}_l$ are closed subspaces, it follows from corollary 2.1 that $DT_{k,l'} \otimes \text{id}$ has closed range. Since DT_k has closed range, the direct sum $DT_k \oplus (DT_{k,l'} \otimes \text{id})$ also has closed range.

Step 2: One can follow the above proof by changing T by $\bar{\partial}$ by use of the complete Kaehler charts. Notice the formula:

$$D\bar{\partial}_J(v) = (DT + J \circ DT \circ i)(v) + N(v)$$

where N involves ∇J , and $N \equiv 0$ when J is integrable. So if it is Kaehler, then $D\bar{\partial}_J = DT + J \circ DT \circ i$ holds. In particular if we decompose the holomorphic local charts as in step 1, then $D\bar{\partial}_J$ can be also expressed as the form $K_1 \oplus (K_2 \otimes \text{id})$. The rest of the argument is parallel to step 1.

This completes the proof.

2.D.2 Index computations: Let $[(M_i, \omega_i, J_i)]$ be a symmetric almost Kaehler sequence. Thus there are holomorphic projections $\pi_k : U_\epsilon(M_k) \rightarrow M_k$ with $\pi_k|_{M_k} = \text{id}$ from small neighbourhoods in $M = \cup_i M_i$.

For $u \in \mathfrak{B}_k$, let $\bar{\pi}_j : \hat{U}(u) \mapsto \mathfrak{B}_j$ be the induced projections for all $j \geq k$.

Lemma 2.7 *Let $[(M_i, \omega_i, J_i)]$ be an almost Kaehler sequence. If $\cup_i \text{Ker} D_u \bar{\partial}_i$ is of finite dimension, then the equality holds:*

$$\text{Ker } D_u \bar{\partial}_J = \cup_i \text{Ker } D_u \bar{\partial}_i.$$

In particular the left hand side is of finite dimension.

Proof : The condition implies $\cup_i \text{Ker } D_u \bar{\partial}_i = \text{Ker } D_u \bar{\partial}_{i_0}$ for some i_0 .

Suppose contrary and assume $\text{Ker } D_u \bar{\partial}_J \neq \cup_i \text{Ker } D_u \bar{\partial}_i$. Let $u_t \subset \hat{U}(u)$ be a smooth curve with $u_0 = u$ and $u'_t|_{t=0} \equiv v \in \text{Ker } D_u \bar{\partial}_J$ but $v \notin \cup_i \text{Ker } D_u \bar{\partial}_i$. Then $D_u \bar{\partial}_j((\pi_j)_*(v)) = (\pi_j)_*(D_u \bar{\partial}(v)) = 0$ vanish for all $j \geq k$. So $(\pi_j)_*(v)$ lies in $\text{Ker } D_u \bar{\partial}_j$. It must be contained in $\text{Ker } D_u \bar{\partial}_{i_0}$ by the assumption. Since j is arbitrary, this implies $v \in \text{Ker } D_u \bar{\partial}_{i_0}$.

This completes the proof.

Let us denote $\bar{\partial}_J : \hat{U}(u) \mapsto \mathfrak{E}|\hat{U}(u)$ and $\bar{\partial}_i : \mathfrak{B}_i \mapsto \mathfrak{E}_i$ respectively.

Proposition 2.2 *Let $[(M_i, \omega_i, J_i)]$ be a symmetric Kaehler sequence. Let us choose any $u \in \mathfrak{M}(M_k, \omega_k, J_k)$.*

If the uniform bounds $\dim \text{Coker } D_u \bar{\partial}_i \leq M$ hold for all $i \geq k$, then $\dim \text{Coker } D_u \bar{\partial}_J \leq M$ also holds.

In particular if it is regular, then it is in fact strongly regular.

Proof : $D \bar{\partial}_J$ has closed range by lemma 2.6. Suppose the estimates $\dim \text{Coker } D_u \bar{\partial}_J \geq M + 1$ could hold, and take orthogonal unit elements u_1, \dots, u_{M+1} in $\text{Coker } D_u \bar{\partial}_J$.

There are large $l \gg k$ so that $u_i^l = (\pi_l)_*(u_i)$ are defined for all $1 \leq i \leq M + 1$. For small $\epsilon > 0$, let us choose sufficiently large l so that the estimates below hold, where $B \subset \text{im } D_u \bar{\partial}_l \subset (\mathfrak{E}_l)_u$ are the unit balls:

$$|u_i^l|^2 \geq 1 - \epsilon, \quad | \langle u_i^l, u_j^l \rangle | \leq \epsilon, \quad | \langle B, u_i^l \rangle | \leq \epsilon.$$

There are numbers $a_1, \dots, a_{M+1} \in \mathbf{R}$ with $\sum_j |a_j|^2 = 1$ so that $v \equiv \sum_i a_i u_i^l$ lie in $\text{im } D_u \bar{\partial}_l$, since $\dim \text{Coker } D_u \bar{\partial}_l \leq M$ hold. Let us pick up i with $|a_i| = \sup_{1 \leq j \leq M+1} |a_j| \geq \frac{1}{\sqrt{M+1}}$. Then one should have the estimates:

$$\epsilon \geq | \langle v, u_i^l \rangle | \geq |a_i|(1 - \epsilon) - \epsilon \sum_{i \neq j} |a_j| \geq |a_i|(1 - \epsilon) - \sqrt{M} \epsilon.$$

Since ϵ are arbitrarily small, this is a contradiction.

This completes the proof.

Example 2.2: \mathbf{CP}^∞ is strongly regular of index 1 by propopsition 2.2.

So for regular and symmetric Kaehler sequences, the moduli spaces of holomorphic curves are strongly regular with the expected indices. With respect to these obseervations, we would like to propose the following:

Conjecture 2.1: Let $[(M_i, \omega_i, J_i)]$ be a symmetric Kaehler sequence.

(1) One can perturb the complex structure (to be almost Kaehler) so that the result could become strongly regular.

(2) For irregular case, index $D \bar{\partial}_J = M$ hold when index $D \bar{\partial}_i = M$ for all i and $\cup_i \text{Ker } D \bar{\partial}_i$ is of finite dimension.

(3) Suppose moreover it is regular (and hence strongly regular). Then the embedding:

$$\mathfrak{M}([(M_i, \omega_i, J_i)]) = \cup_i \mathfrak{M}_i(M_i, \omega_i, J_i) \subset \hat{\mathfrak{M}}([(M_i, \omega_i, J_i)])$$

is in fact equality.

2.D.3 Compactness of moduli spaces: Let $[(M_i, \omega_i, J_i)]$ be an isotropic symmetric almost Kaehler sequence.

Lemma 2.8 *Suppose that each connected component of $\mathfrak{M}([(M_i, \omega_i, J_i)])$ is bounded. Then the equality holds:*

$$\mathfrak{M}([(M_i, \omega_i, J_i)]) = \mathfrak{M}(M_0, \omega_0, J_0).$$

In particular if it is minimal, then $\mathfrak{M}([(M_i, \omega_i, J_i)])$ is compact.

Notice that the condition is satisfied for regular almost Kaehler sequences whose moduli spaces have 1 dimensional.

Proof: Let us choose an element $[u] \in \mathfrak{M}([(M_i, \omega_i, J_i)])$. By the assumption there is some l_0 so that the connected component $\mathfrak{M}(u)$ containing u has all their images in M_{l_0} .

Suppose there could exist some k with its symmetry over $(M_k, M_{l'})$ for $l' = l(k) > k$, such that $\mathfrak{M}(u)$ has all their images in $M_{l'}$ but not all in M_k .

Let $P_i^t : (M, W_i, M_k) \cong (M, M_{l'}, M_k)$ be the isotropies for $i \geq l'$, where $M = \cup_i M_i$. There is some $u' \in \mathfrak{M}(u)$ so that the images of $P_i^0(u') = u'$ are contained in $M_{l'}$, while $(P_i^1)^{-1}(u')$ are not the case for all $i > l'$. This implies that the images of $\mathfrak{M}(u)$ cannot be contained in $M_{l'}$, since $(P_i^1)^{-1}(u')$ must be contained in $\mathfrak{M}(u)$. This contradicts to the assumption. So $\mathfrak{M}(u)$ must be contained in M_k .

Next let us replace the pair $(k, l'(k))$ by $(k-1, l'(k-1))$. Because of the relation $l'(k-1) > k-1$, the inequality $l'(k-1) \geq k$ must hold. In particular the images of $\mathfrak{M}(u)$ are contained in $M_{l'(k-1)}$, and proceed the same argument. Then we find that $\mathfrak{M}(u)$ is contained in M_{k-1} .

Let us continue this process. Then finally we find that the images of $\mathfrak{M}(u)$ must be contained in M_0 . This completes the proof.

It would be interesting to study what happens for the cases of positive dimension. We would like to propose the following:

Conjecture 2.2: Suppose $\mathfrak{M}([(M_i, \omega_i, J_i)])$ is a smooth manifold of finite

dimension. Then the equality:

$$\mathfrak{M}([(M_i, \omega_i, J_i)]) = \mathfrak{M}(M_k, \omega_k, J_k)$$

holds for some k .

Notice that the strong regularity condition is stable under small perturbations, and we expect that such property can be studied over deformations of these sequences. This is the topic at the next section.

3 Infinitesimal neighbourhoods of almost Kaehler sequences

One of the quite characteristic properties which infinite dimensional spaces possess, is that they can contain many spaces as their proper subsets which can include even themselves. In our formulation, infinite dimensional spaces are consisted by sequences of finite dimensional spaces, which leads us canonically to introduce *neighbourhoods* of such sequences as some sets of another sequences. In the finite dimensional case, notion of neighbourhoods will require some dimension restrictions. However such limitations are free for our infinite dimensional sequences.

In later sections, we apply such notions to study *stability* of the invariants of almost Kaehler sequences under “very small perturbations”, which measure continuity of these invariants in the framework of the neighbourhoods.

3.A Convergence: Let us start from the finite dimensional case. Let $\{X_i\}_{i=1}^{\infty}$ be a family of smooth manifolds of the same dimension embedded into another finite dimensional smooth manifold M .

We say that the set $\{X_i\}$ *converges* to X in M , if there exist coverings of $X = \cup_l U_l$ and $X_i = \cup_l U_l^i$ so that for all sufficiently large $i \gg 0$, there are diffeomorphisms $F_l^i : U_l \cong U_l^i (\subset M)$ which converge to the identity in C^∞ topology in M . Notice that if X_i are sufficiently near X in C^∞ topology, then they are isotopic to X .

3.A.1 Convergence of almost Kaehler sequences: Let us introduce the following:

Definition 3.1 Let $[(M_i, \omega_i, J_i)]$ be an almost Kaehler sequence. Let us say that a family of almost Kaehler sequences $\{[(M_i^l, \omega_i^l, J_i^l)]\}_{l=1}^\infty$ converges to $[(M_i, \omega_i, J_i)]$, if there are positive $\epsilon > 0$, subindices $\{k(i)\}_i$ with $k(i) \geq i$ and compatible C^∞ embeddings of $[(M_i^l, \omega_i^l, J_i^l)]$ into $M = \cup_k M_k$ for all l :

$$I(i, l) : M_i^l \hookrightarrow M_{k(i)}, \quad I(i+1, l)|_{M_i^l} = I(i, l)$$

with almost complex $J(i, l)$ and symplectic $\omega(i, l)$ structures on the open ϵ tubular neighbourhoods $I(i, l)(M_i^l) \subset U(i, l) \subset M$, which extend the given ones:

$$J(i, l)|_{I(i, l)(M_i^l)} = J_i^l, \quad \omega(i, l)|_{I(i, l)(M_i^l)} = \omega_i^l$$

so that the following three conditions hold:

(1) For each l , there is a uniformly bounded covering $\{(p, \varphi(p))\}$ on $[(M_i^l, \omega_i^l, J_i^l)]$, and its extension $\{(p, \psi(p))\}$ by ϵ complete almost Kaehler charts over $(U(i, l), \omega(i, l), J(i, l))$ for all $p \in I(i, l)(M_i^l)$:

$$\begin{aligned} \varphi(p) : \cup_{s \geq 1} D^{2s}(\epsilon) &\hookrightarrow \text{im } \varphi(p) \subset \cup_{r \geq 1} M_r^l \\ &\cap \quad \cap \\ \psi(p) : \cup_{s \geq 1} D^{2s}(\epsilon) &\hookrightarrow U(i, l) \subset \cup_{r \geq 1} M_r \end{aligned}$$

(2) There are families of holomorphic maps:

$$\pi_i^l : U(i, l) \mapsto M_i^l$$

with respect to $\omega(i, l)$, which satisfy the following properties:

$$\pi_i^l|_{M_i^l} = \text{id}, \quad \pi_i^l(\psi(p)(m)) = \psi(p)(\pi_i^l(m))$$

for all $m \in U(i, l)$, where $\pi_i^l : \cup_{s \geq 1} D^{2s}(\epsilon) \rightarrow D^{2d_i^l}(\epsilon)$ are the projections with $d_i^l = \dim M_i^l$.

(3) All derivatives of the operators converge to zero as $l \rightarrow \infty$:

$$\sup_i \|\nabla^\alpha(J(i, l) - J)\|_g U(i, l), \quad \sup_i \|\nabla^\alpha(\omega(i, l) - \omega)\|_g U(i, l) \rightarrow 0.$$

(4) For each i , $\{I(i, l)(M_i^l)\}_{l \geq 1} \subset M_{k(i)}$ converges to M_i in $M_{k(i)}$ in C^∞ .

Notice that in general $I(i, l)(M_i^l)$ may be far away from M_i as $i, l \rightarrow \infty$ (convergence in (3) is not assumed uniform with respect to i).

In short, we will denote convergence by the notation:

$$\{[(M_i^l, \omega_i^l, J_i^l)]\}_l \rightarrow [(M_i, \omega_i, J_i)].$$

We say that $\{[(M_i^l, \omega_i^l, J_i^l)]\}_l \rightarrow [(M_i, \omega_i, J_i)]$ is *minimal convergence*, if all almost Kaehler sequences are minimal.

Examples 3.1: (1) Let us choose a uniform family of almost Kaehler manifolds (X_i, ω_i, J_i) , $i = 1, 2, \dots$, and take another family of almost Kaehler manifolds (X_0, ω_0^l, J_0^l) so that both of ω_0^l and J_0^l converge to ω_0 and J_0 in C^∞ as $l \rightarrow \infty$ respectively. Let us put:

$$(M_i^l, \omega_i^l, J_i^l) \equiv (X_0 \times \dots \times X_i, \omega_0^l + \omega_1 + \dots + \omega_i, J_0^l \oplus J_1 \oplus \dots \oplus J_i).$$

Then $\{[(M_i^l, \omega_i^l, J_i^l)]\}$ is convergent to the product with $I(i, l) = \text{id}$.

(2) Let (X, ω, J) be an almost Kaehler manifold with a fixed point $x_0 \in X$. Let $\omega(i) = \omega + \dots + \omega$ and $J(i) = J \oplus \dots \oplus J$ be almost Kaehler data on $X \times \dots \times X$. We put $M_i = \times^i X$ where $M_i = \times^i X \times \{x_0\} \subset M_{i+1}$.

Let $a : \mathbf{N} \mapsto \mathbf{N}$ be any proper function. Let us choose all the same $M_i^l = (M_i, \omega(i), J(i))$, but take embeddings $I(i, l) : M_i^l \hookrightarrow M_{i+1}$ as:

$$I(i, l)((m_0, \dots, m_i)) = (m_1, \dots, m_{a(l)}, x_0, m_{a(l)+1}, \dots, m_i).$$

Then $\{[(M_i^l, \omega(i), J(i))]\}$ is a convergent family.

3.B Infinitesimal neighbourhoods: Let us take an almost Kaehler sequence $[(M_i, \omega_i, J_i)]$, and two families of almost Kaehler sequences:

$$\{[(M_i^l, \omega_i^l, J_i^l)]\}_{l \geq 1}, \quad \{[(N_i^l, \tau_i^l, I_i^l)]\}_{l \geq 1} \rightarrow [(M_i, \omega_i, J_i)]$$

which both converge to the same almost Kaehler sequence.

Let us say that the two families are *equivalent*, if there are infinite subindices $\{k(l)\}_l$ and $\{k'(l)\}_l$ such that there are isomorphisms between almost Kaehler sequences for all l :

$$[(M_i^{k(l)}, \omega_i^{k(l)}, J_i^{k(l)})]_{i \geq 1} \cong [(N_i^{k'(l)}, \tau_i^{k'(l)}, I_i^{k'(l)})]_{i \geq 1}.$$

We denote the equivalence class of $\{[(M_i^l, \omega_i^l, J_i^l)]\}_{l \geq 1}$ by $[[(M_i^l, \omega_i^l, J_i^l)]]$.

Definition 3.2 *An infinitesimal neighbourhood of $[(M_i, \omega_i, J_i)]$ is given by the set of equivalence classes by convergent families:*

$$\mathfrak{N}([(M_i, \omega_i, J_i)]) = \{[(M_i^l, \omega_i^l, J_i^l)] : \{[(M_i^l, \omega_i^l, J_i^l)]\}_{l \geq 1} \rightarrow [(M_i, \omega_i, J_i)]\}.$$

Let $[(M_i, \omega_i, J_i)]$ and $\{[(M_i^l, \omega_i^l, J_i^l)]\}_{l \geq 1}$ be minimal almost Kaehler sequences, and suppose the family converges to $[(M_i, \omega_i, J_i)]$ with the data $\{I(i, l) : M_i^l \hookrightarrow M_{k(i)}\}_{i, l}$. We say the convergence *keeps* minimal classes, if $I(l)_*$ maps the minimal classes to the one over $[(M_i, \omega_i, J_i)]$ on π_2 .

Now by restricting on minimal almost Kaehler sequences, we define the infinitesimal neighbourhoods of minimal almost Kaehler sequences:

$$\begin{aligned} \mathfrak{N}^m([(M_i, \omega_i, J_i)]) &= \{[(M_i^l, \omega_i^l, J_i^l)] : \{[(M_i^l, \omega_i^l, J_i^l)]\}_{l \geq 1} \rightarrow [(M_i, \omega_i, J_i)] \\ &\quad \{[(M_i^l, \omega_i^l, J_i^l)]\}_{l \geq 1} : \text{minimal families which keep minimal classes} \}. \end{aligned}$$

3.C Moduli theory over perturbations of spaces: So far we have studied global analysis of the moduli spaces of holomorphic curves into infinite dimensional spaces. It turned out that their behaviours are well controlled if we assume integrability with high symmetry, where they are stable under small perturbations of their structures under such situations. This is the key aspect which allows us to study moduli theory over infinitesimal neighbourhoods.

Let $[(M_i, \omega_i, J_i)]$ be an almost Kaehler sequence. Let us consider its infinitesimal neighbourhood and take an element:

$$[(M_i^l, \omega_i^l, J_i^l)] \in \mathfrak{N}^m([(M_i, \omega_i, J_i)]).$$

The following holds since regularity condition is open:

Lemma 3.1 *Let $[(M_i, \omega_i, J_i)]$ be a minimal and regular almost Kaehler sequence. Suppose $\mathfrak{M}([(M_i, \omega_i, J_i)])$ is bounded and S^1 freely 1 dimensional manifold. Then there is $l_0 \gg 0$ so that there are S^1 compatible embeddings for all $l \geq l_0$:*

$$\mathfrak{M}([(M_i, \omega_i, J_i)]) \subset \mathfrak{M}([(M_i^l, \omega_i^l, J_i^l)]).$$

Proof: It is enough to use well known analysis which are applied for finite dimensional almost Kaehler manifolds. By the assumption, there is i so that $\mathfrak{M}([(M_i, \omega_i, J_i)]) = \mathfrak{M}(M_i, \omega_i, J_i)$ are both compact. Let us consider a family of embeddings $I(i, l) : M_i^l \hookrightarrow M_k$ in definition 3.1.

By compactness there is a large l_0 so that for all $l \geq l_0$, the images of any $u \in \mathfrak{M}(M_i, \omega_i, J_i)$ are contained in $U(i, l) \subset \cup_j M_j$. Then by use of the projections $\pi_i^l : U(i, l) \rightarrow M_i^l$, let us consider a smooth maps $u' \equiv \pi_i^l \circ u : \mathbf{CP}^1 \rightarrow M_i^l$. The differentials of the Cauchy-Riemann operators must be surjective at u' since $(M_i^l, \omega_i^l, J_i^l)$ converge to (M_i, ω_i, J_i) smoothly. Let us apply the infinite dimensional implicit function theorem to u' to obtain a holomorphic curve u'' with respect to $(M_i^l, \omega_i^l, J_i^l)$. This assignment $u' \rightarrow u''$ extends to the S^1 freely equivariant ones. This completes the proof.

Notice that we have used compactness of the moduli spaces so that the inverse of the differentials of C-R operators satisfy uniform estimates from above.

In order to obtain the converse embedding, we use the strong regularity condition below.

For any map $u : \mathbf{CP}^1 \rightarrow M_i^l$, let us consider the compositions:

$$v \equiv I(i, l) \circ u : \mathbf{CP}^1 \rightarrow M_{k(i)}$$

with the embeddings $I(i, l) : M_i^l \hookrightarrow M_{k(i)}$. There is a unique $a > 0$ so that the translation $v'(m) = v(am)$ on $\mathbf{C} \subset \mathbf{CP}^1$ satisfies the condition $\int_{D(1)} (v')^*(\omega_{k(i)}) = \frac{1}{2} < \omega, \alpha >$ as in 2.A. This gives the embedding:

$$\mathfrak{B}(M_i^l, \omega_i^l, J_i^l) \subset \mathfrak{B}(M_{k(i)}, \omega_{k(i)}, J_{k(i)}).$$

Notice that if l is sufficiently large, then this is very near just the induced maps by the composition with $I(i, l)$, namely a above is near 1.

Let us consider a holomorphic curve $u \in \mathfrak{M}([(M_i^l, \omega_i^l, J_i^l)])$, and regard:

$$u \in \mathfrak{B}([(M_i, \omega_i, J_i)]).$$

As in 2.C, let $\hat{U}(u)$ be the Hilbert completion of the open subset $U(u)$ in $\mathfrak{B}([(M_i, \omega_i, J_i)])$, and consider $\bar{\partial}_J : \hat{U}(u) \mapsto \hat{\mathfrak{E}}|\hat{U}(u)$ with the differential $D\bar{\partial}_J : T_u \hat{U}(u) \mapsto \hat{\mathfrak{E}}_u$.

Definition 3.3 $\mathfrak{N}^m([(M_i, \omega_i, J_i)])$ is strongly regular, if for any element $[(M_i^l, \omega_i^l, J_i^l)]$, there is l_0 and $C > 0$ so that for all $l \geq l_0$ and for any $u \in \mathfrak{M}([(M_i^l, \omega_i^l, J_i^l)])$:

$$D\bar{\partial}_J : T_u\hat{U}(u) \mapsto \hat{\mathfrak{E}}_u$$

is surjective over $[(M_i, \omega_i, J_i)]$. Moreover the linear inverse $(D\bar{\partial}_u)^{-1}$ satisfies the uniform estimate:

$$C \geq |(D\bar{\partial}_u)^{-1}|.$$

Notice that $[(M_i, \omega_i, J_i)]$ must be strongly regular, while $[(M_i^l, \omega_i^l, J_i^l)]$ might not be strongly regular.

Proposition 3.1 Let $[(M_i, \omega_i, J_i)]$ be a minimal almost Kaehler sequence.

Suppose:

- (1) the moduli space $\mathfrak{M}([(M_i, \omega_i, J_i)])$ is S^1 freely 1 dimensional, and
- (2) $\mathfrak{N}^m([(M_i, \omega_i, J_i)])$ is strongly regular.

Then for any element $[(M_i^l, \omega_i^l, J_i^l)] \in \mathfrak{N}^m([(M_i, \omega_i, J_i)])$, there is $l_0 \gg 0$ so that the S^1 equivariant homeomorphisms:

$$\mathfrak{M}([(M_i^l, \omega_i^l, J_i^l)]) \cong \mathfrak{M}([(M_i, \omega_i, J_i)])$$

hold for all $l \geq l_0$.

Proof : Combining with lemma 3.1, it is enough to construct the S^1 equivariant embeddings $\mathfrak{M}([(M_i^l, \omega_i^l, J_i^l)]) \hookrightarrow \mathfrak{M}([(M_i, \omega_i, J_i)])$ for all $l \geq l_0$.

Let us take $u \in \mathfrak{M}(M_i^l, \omega_i^l, J_i^l)$. By lemma 2.2, there are constants c_α independent of u so that the pointwise estimates $|\nabla^\alpha u| \leq c_\alpha$ hold.

Let us regard $u \in \mathfrak{B}(M_{k(i)}, \omega_{k(i)}, J_{k(i)})$, and let $\hat{U}(u)$ be the Hilbert manifolds over $\cup_i M_i$ as above. It follows from the uniform estimates that there is small $\delta > 0$ independent of u so that δ ball $B_\delta(u) \subset T_u\hat{U}(u)$ can be regarded as an open subset in $\hat{U}(u)$.

It follows from the strong regularity condition that for any element $[(M_i^l, \omega_i^l, J_i^l)] \in \mathfrak{N}^m([(M_i, \omega_i, J_i)])$, there is $l_0 \gg 0$ so that the S^1 freely equivariant embeddings:

$$\Phi : \hat{\mathfrak{M}}([(M_i^l, \omega_i^l, J_i^l)]) \hookrightarrow \hat{\mathfrak{M}}([(M_i, \omega_i, J_i)])$$

are given into the smooth manifolds for all $l \geq l_0$ (see above 2.C.2).

Let us denote the holomorphic projection $\pi_k : \hat{U}(u) \mapsto \mathfrak{B}(M_k, \omega_k, J_k)$. Then $v \equiv \pi_k(\Phi(u))$ must satisfy $\bar{\partial}_J(v) = 0$ for all large k . It implies the equality $v = \Phi(u)$ for some large k , since $\mathfrak{M}([(M_i, \omega_i, J_i)])$ are 1 dimensional smooth manifolds, and the injectivity radii on $\mathfrak{M}([(M_i, \omega_i, J_i)]) \subset \mathfrak{B}([(M_i, \omega_i, J_i)])$ are uniformly bounded from below by uniform surjectivity of $D\bar{\partial}_J$. This completes the proof.

3.C.2 Strong regularity on infinitesimal neighbourhoods: We have introduced a notion of quasi transitivity in 1.B.4. Such property will be satisfied if the automorphism group is sufficiently large. For example \mathbf{CP}^∞ is the case. In general, regularity on an almost Kaehler sequence does not imply the strong one on its infinitesimal neighbourhoods. In order to guarantee such property, we will require high symmetry and integrability.

Proposition 3.2 *Let $[(M_i, \omega_i, J_i)]$ be a minimal and quasi-transitive almost Kaehler sequence. Suppose it is strongly regular.*

Then $\mathfrak{N}^m([(M_i, \omega_i, J_i)])$ is also strongly regular.

Proof: Let us use the notations in 3.A.1. Let $U(k, l) \subset M \equiv \cup_i M_i$ be the open neighbourhoods of $I(k, l)(M_k^l)$, and extend ω_k^l and J_k^l as $\omega(k, l)$ and $J(k, l)$ over $U(k, l)$ respectively.

Step 1: We verify that for any $i > 0$, there is a large $l_0 = l(i)$ so that $(U(i, l), \omega(i, l), J(i, l))$ is strongly regular at u for any holomorphic curve $u \in \mathfrak{M}(M_i^l, \omega_i^l, J_i^l)$ for all $l \geq l_0$.

We claim that there exists $v \in \mathfrak{M}(M_{k(i)}, \omega_{k(i)}, J_{k(i)})$ which is sufficiently near u in $\mathfrak{B}([(M_i, \omega_i, J_i)])$. Suppose contrary. Then there exist $\epsilon > 0$ and $u_l \in \mathfrak{M}(M_i^l, \omega_i^l, J_i^l)$ so that ϵ neighbourhoods $B_\epsilon(u_l) \subset \mathfrak{B}([(M_i, \omega_i, J_i)])$ contain no elements in $\mathfrak{M}(M_{k(i)}, \omega_{k(i)}, J_{k(i)})$. However by lemma 2.2, a subsequence must converge to a solution in $\mathfrak{M}(M_{k(i)}, \omega_{k(i)}, J_{k(i)})$. This verifies the claim.

Since $(U(k, l), \omega, J)$ and $(U(k, l), \omega(k, l), J(k, l))$ are uniformly near and since strong regularity is an open condition, this implies that the latter is strongly regular at u .

Step 2: Let us fix sufficiently large $N > 0$ and N points $m_0, \dots, m_{N-1} \in S^2$ with $m_0 = 0, m_1 = \infty$. $d(u(m_i), u(m_j))$ are uniformly bounded for any $u \in \mathfrak{M}([(M_i^l, \omega_i^l, J_i^l)])$ by lemma 2.2.

Let us use the condition of quasi transitivity, and choose an automorphism A on $M = \cup_i M_i$ with respect to $\{p_i = u(m_i)\}_{i=0}^{N-1}$ as in 1.B.4, so that $A(p_i) \in M_{k_0}$ for some k_0 .

Strong regularity at u is equivalent to that on $A \circ u$, which follows from uniformity of complete local charts. Thus by replacing u by $A \circ u$, one may assume $u(m_i) \in M_{k_0} \subset M = \cup_i M_i$.

There is small $\epsilon > 0$ independent of u so that the bounds $d(M_{k_0}, u(m)) < \epsilon$ hold at any $m \in S^2$ again by lemma 2.2. Let $U_\epsilon(M_{k_0}) \subset M = \cup_i M_i$ be the ϵ neighbourhood, and $\pi_{k_0} : U_\epsilon(M_{k_0}) \mapsto M_{k_0}$ be the holomorphic projection. Then by composition, the projection $u' = \pi_{k_0}(u) : S^2 \mapsto M_{k_0}$ is a small deformation of u , where the uniform estimates:

$$\|u - u'\|C^\alpha < \epsilon_\alpha$$

hold by lemma 2.2. Here ϵ_α are independent of i, l of $\mathfrak{M}([(M_i^l, \omega_i^l, J_i^l)])$.

Let $\bar{\partial}_J$ be the C-R operator with respect to $[(M_i, \omega_i, J_i)]$. Then there is a small constant $\delta > 0$ with $\|\bar{\partial}_J(u')\| < \delta$.

Step 3: We show that l_0 in step 1 can be chosen independently of choice of i . This is enough to verify the proposition.

Let us choose $u_l \in \mathfrak{M}([(M_i^l, \omega_i^l, J_i^l)])$ and put $u'_l = \pi_{k_0}(u_l)$. By step 2, these family admit uniformly bounded derivatives, and $\|\bar{\partial}_J(u'_l)\|$ converges to zero uniformly as $l \rightarrow \infty$. So a subsequence of u'_l converges to some u as $l \rightarrow \infty$ and $\bar{\partial}_J(u) = 0$ holds. This implies that u'_l are strongly regular on $[(M_i, \omega_i, J_i)]$ for all large $l \geq l_0$ where l_0 are independent of i . Then u_l are also the same by the above uniform estimates.

This completes the proof.

Example 3.1: $\mathfrak{N}^m([(CP^i, \omega_i, J_i)])$ is strongly regular.

Combining with proposition 3.1 and 3.2, we obtain the following:

Corollary 3.1 *Let $[(M_i, \omega_i, J_i)]$ be a minimal, quasi-transitive and strongly regular almost Kaehler sequence.*

If the moduli space $\mathfrak{M}([(M_i, \omega_i, J_i)])$ is S^1 freely 1 dimensional, then for any element $[(M_i^l, \omega_i^l, J_i^l)] \in \mathfrak{N}^m([(M_i, \omega_i, J_i)])$, there is $l_0 \gg 0$ so that the S^1 equivariant homeomorphisms:

$$\mathfrak{M}([(M_i^l, \omega_i^l, J_i^l)]) \cong \mathfrak{M}([(M_i, \omega_i, J_i)])$$

are given for all $l \geq l_0$.

Now we collect the previous results which we have induced so far:

Theorem 3.1 *Let $[(M_i, \omega_i, J_i)]$ be a regular and minimal almost Kaehler sequence.*

(1) *Suppose it is isotropic and symmetric. If each connected component of the moduli space is bounded, then the equality holds:*

$$\mathfrak{M}([(M_i, \omega_i, J_i)]) = \mathfrak{M}(M_0, \omega_0, J_0).$$

In particular if the moduli space is 1 dimensional, then the above equality holds, which are both compact.

(2) *If it is symmetric Kaehler, then it is strongly regular.*

If moreover it is quasi-transitive, then $\mathfrak{N}^m([(M_i, \omega_i, J_i)])$ is strongly regular.

(3) *Under all the conditions in (1) and (2), it follows that for any element $[(M_i^l, \omega_i^l, J_i^l)] \in \mathfrak{N}^m([(M_i, \omega_i, J_i)])$, there is $l_0 \gg 0$ so that the S^1 equivariant homeomorphisms:*

$$\mathfrak{M}([(M_i^l, \omega_i^l, J_i^l)]) \cong \mathfrak{M}([(M_i, \omega_i, J_i)])$$

are given for all $l \geq l_0$.

4 Application to Hamiltonian dynamics

In this section we apply theory of moduli spaces we developed so far to study of Hamiltonian dynamics defined by smooth and bounded functions over almost Kaehler sequences.

4.A Capacity invariant: In Hamiltonian dynamics, capacity invariant

contains deep information on the periodic solutions and has been playing one of the central roles over finite dimensional symplectic manifolds.

Let (M, ω) be a closed symplectic manifold of finite dimension. We say that a smooth function $f : M \mapsto [0, \infty)$ is Hamiltonian. The *Hamiltonian vector field* X_f is uniquely defined by the relation:

$$df(\cdot) = \omega(\cdot, X_f).$$

A *periodic solution* $x : [0, T] \rightarrow M$ with $x(0) = x(T)$ satisfies the equation:

$$\dot{x} = X_f(x)$$

and $T \geq 0$ is called *period*.

Remark 4.1: Let $a > 0$ be a positive number and $\tilde{f}(x) = af(x)$ be the function multiplied by a . If $x : [0, T] \rightarrow M$ is a periodic solution to f with period T , then $\tilde{x} : [0, a^{-1}T] \rightarrow M$ given by $\tilde{x}(t) = x(at)$ is also a periodic solution to \tilde{f} , since the equality $X_{\tilde{f}} = aX_f$ holds. So its dynamical properties are scaling invariant under multiplication by positive numbers. Moreover the ‘height’ of functions is in inverse proportion to periods.

Let us introduce an invariant over symplectic manifolds with respect to periodic solutions ([HZ]). A Hamiltonian function is *pre admissible* if there are open sets $U, V \subset M$ with $f|_U \equiv c = \sup f$ and $f|_V \equiv 0$. Moreover it is *admissible* if in addition, any periodic solution is either constant or period $T = T(x) > 1$, where $x : [0, T] \rightarrow M$ with $x(0) = x(T)$, and X_f is the Hamiltonian vector field.

Let us denote the set of admissible functions by $H_a(M, \omega)$. Then we define the *capacity* of (M, ω) by the following:

$$c(M, \omega) = \sup\{m(f) \equiv \sup f - \inf f \geq 0 : f \in H_a(M, \omega)\}.$$

Let N be a symplectic manifold with boundary. $f : N \mapsto [0, \infty)$ is pre admissible, if there is an open set U with $f|_U \equiv c = \sup f$ and it vanishes on some neighbourhood of boundary. It is admissible if in addition, any periodic solution $\dot{x} = X_f(x)$ is either constant or period > 1 . We define the capacity by the same way over symplectic manifolds with boundary.

This numerical invariant satisfies some axioms of capacity. Notice that admissible functions always exist, since the Darboux's chart exists at any point, and $c(D, \omega) = \pi$ where (D, ω) is the standard symplectic disk.

Let us consider the upper bounds of the invariants. Let (M, ω, J) be a minimal almost Kahler manifold of finite dimension, and let us fix a minimal element $\alpha \in \pi_2(M)$.

Let us state a basic relation between periodic solutions and holomorphic curves:

Lemma 4.1 (HV) *Let $f : M \rightarrow [0, \infty)$ be a pre-admissible function. Then there are non trivial periodic solutions whose periods satisfy the estimates*

$$T < m(f)^{-1}m$$

where m is the minimal invariant, if the moduli space of holomorphic curves with respect to α is non empty, regular, 1 dimensional and S^1 freely cobordant to non zero.

In particular under the above conditions, the estimates:

$$c(M, \omega) \leq m$$

hold. This is used to estimate capacity invariants over almost Kaehler sequences defined below.

4.B Asymptotic periodic solutions: Let $[(M_i, \omega_i, J_i)]$ be an almost Kaehler sequence. Later on we fix a uniformly bounded covering by $\epsilon > 0$ complete almost Kaehler charts.

Let $f : M \mapsto \mathbf{R}_+$ be a bounded Hamiltonian function and put $f_i = f|_{M_i}$. A family of smooth loops $x_i : [0, T_i] \mapsto M_i$ with $x_i(0) = x_i(T_i)$ is a periodic solution over $[(M_i, \omega_i, J_i)]$, if these satisfy the equations:

$$\dot{x}_i(t) = X_{f_i}(x_i(t))$$

for all $0 \leq t \leq T_i$ and $i = 0, 1, \dots$. We will denote such a family by $[x_i]$.

Let $[(M_i^l, \omega_i^l, J_i^l)] \in \mathfrak{N}([(M_i, \omega_i, J_i)])$ be an element of the infinitesimal neighbourhood. Let us consider a family of bounded Hamiltonians:

$$f_l : M^l \equiv \cup_i M_i^l \rightarrow \mathbf{R}$$

and take a family of periodic solutions $[x_i^l]$ over $[(M_i^l, \omega_i^l, J_i^l)]$ with their periods T_i^l . Passing through the embeddings $I(i, l) : M_i^l \hookrightarrow M_{k(i)}$, one may regard these loops as:

$$x_i^l : [0, T_i^l] \rightarrow M_{k(i)}.$$

Let $\pi_i : U_\epsilon(M_i) \rightarrow M_i$ be the holomorphic projections.

Proposition 4.1 *Suppose $[(M_i, \omega_i, J_i)]$ is quasi transitive, and both the uniform bounds hold for all $\alpha \geq 0$:*

$$\sup_l \|f_l\| C^\alpha(M^l) \leq C_\alpha, \quad \sup_{i,l} T_i^l \leq T < \infty.$$

Then there is a bounded Hamiltonian $g : M \equiv \cup_i M_i \rightarrow \mathbf{R}$ and a family of automorphisms A_i over (M, ω, J) so that for some l_i , the family of loops $z_i \equiv \pi_i(A_i \circ x_i^{l_i}) : [0, T_i^{l_i}] \rightarrow M_i$ satisfy the following asymptotics:

$$\lim_{i \rightarrow \infty} \sup_t |\dot{z}_i - X_{g_i}(z_i)|(t) = 0.$$

Proof: Let us recall the notations in 3.A.1.

Step 1: Let us choose a sufficiently large N and N points $0 \leq t_i = \frac{T}{N-1}i \leq T$ for $i = 0, 1, \dots, N-1$. By quasi transitivity, there are some k and automorphisms A_i^l over $[(M_i, \omega_i, J_i)]$ so that:

$$y_i^l(t_i) \equiv A_i^l(x_i^l(t_i)) \in M_k$$

hold for all i . By the assumption of the uniform bounds, there is small $\delta > 0$ with the bounds in M for all $0 \leq t \leq T_i^l$:

$$d(y_i^l(t), M_k) \leq \delta.$$

Step 2: M_i^l admit the embeddings $I(i, l) : M_i^l \hookrightarrow M$, and there are some open subsets:

$$I(i, l)(M_i^l) \subset U(i, l) \subset M$$

where $U(i, l)$ contain ϵ neighbourhoods of $I(i, l)(M_i^l)$ for some $\epsilon > 0$.

Let us fix i , and consider the restrictions $f_l|_{M_i^l}$ and regard them as:

$$f_l : I(i, l)(M_i^l) \rightarrow \mathbf{R}.$$

We extend them as $f_i^l : M \rightarrow \mathbf{R}$ as follows. Let $\pi_i^l : U(i, l) \rightarrow I(i, l)(M_i^l)$ be the holomorphic projections. Let $\mu : [0, \epsilon) \rightarrow [0, 1]$ be a cut off function with $\mu(m) = 0$ for all $m \geq \frac{1}{2}\epsilon$ and $\mu \equiv 1$ near 0. Then we put:

$$f_i^l(m) \equiv \mu(d(m, M_i^l)) f_l(\pi_i^l(m))$$

for $m \in U(i, l) \subset M$, and $f_i^l|_{M \setminus U(i, l)} \equiv 0$. Let us put $h_i^l \equiv f_i^l \circ (A_i^l)^{-1}$. These families satisfy the following properties:

- (1) $\{h_i^l\}_{i,l}$ are uniformly bounded as $\|h_i^l\|_{C^\alpha(M)} \leq C_\alpha$ for all α .
- (2) For $\bar{h}_i^l \equiv h_i^l|_{M_i}$, $x_i^l : [0, T_i^l] \rightarrow M_{k(i)}$ satisfy the asymptotics:

$$\lim_{l \rightarrow \infty} \sup_t |\dot{y}_i^l(t) - X_{\bar{h}_i^l}(y_i^l(t))| = 0.$$

- (3) For the projections $\pi_i : U_\epsilon(M_i) \rightarrow M_i$,

$$\lim_{i \rightarrow \infty} \|y_i^l - \pi_i(y_i^l)\|_{C^1} = 0.$$

Step 3: A subsequence of the family $\{h_i^l\}_i$ converges weakly to a bounded Hamiltonian g so that for $z_i \equiv \pi_i(y_i^l)$, the asymptotics hold:

$$\lim_{l \rightarrow \infty} \sup_t |\dot{z}_i(t) - X_g(z_i(t))| = 0$$

by step 2 and lemma 1.3. This completes the proof.

In the context of this paper, asymptotic analysis play one of the central roles in studying increasing sequences of manifolds. Proposition 4.1 presents a motivation quite naturally to introduce some families of loops which approach periodic solutions asymptotically. Conversely asymptotic periodic solutions defined below can be regarded as though a kind of ‘periodic loops’ over the infinitesimal neighbourhoods.

Let $[(M_i, \omega_i, J_i)]$ be an almost Kaehler sequence. Let $[x_i]$ be a family of smooth loops:

$$x_i : [0, T_i] \mapsto M_i$$

with $x_i(0) = x_i(T_i)$ and $i = 0, 1, \dots$

We say that $[x_i]$ is an *asymptotic loop*, if $T = \limsup_i T_i < \infty$ is finite. We call T as the *periods* of the family.

An asymptotic loop $[x_i]$ is *small*, if $0 \leq \limsup_i T_i \leq 1$ holds.

We say that an asymptotic loop $[x_i]$ is *non trivial*, if uniform estimates:

$$\text{length } x_i \equiv \int_0^{T_i} |\dot{x}_i(t)| dt \geq \delta > 0$$

hold for all large $i \geq i_0 = i_0([x_i])$ and some positive $\delta = \delta([x_i]) > 0$.

Definition 4.1 Let $f : M \mapsto \mathbf{R}_+$ be a bounded Hamiltonian function and put $f_i = f|_{M_i}$. An asymptotic periodic solution is an asymptotic loop $[x_i]$ satisfying:

$$\sup_t |\dot{x}_i - X_{f_i}(x_i)|(t) \rightarrow 0, \quad i \rightarrow \infty.$$

Remark 4.2: (1) Trivial asymptotic periodic solutions $[x_i]$ with $\text{length } x_i \rightarrow 0$, always exist with any periods, if f attain maximum or minimum values.

(2) Any asymptotic periodic solutions $[x_i]$ must satisfy uniform bounds:

$$\limsup_i \text{length } x_i < \infty.$$

(3) Suppose a family of bounded hamiltonians $\{f^l\}$ over an almost Kaehler sequence converges as $\|f^l - f\|_{C^\alpha(M)} \rightarrow 0$ for all α . Let $\{x_i^l\}_i$ be families of periodic solutions with respect to f^l . Then $y_i \equiv x_i^i$ is an asymptotic periodic solution with respect to f .

4.B.2 Comparison with periodic solutions: Let us compare asymptotic periodic solutions with exact solutions.

Let us say that an asymptotic periodic solution $[x_i]$ is *exact*, if x_i are periodic solutions:

$$\dot{x}_i - X_{f_i}(x_i) = 0$$

to the restrictions $f|_{M_i}$ for infinitely many i .

There are almost Kaehler sequences and bounded Hamiltonians over them so that there are no exact periodic solutions but do exist asymptotic periodic solutions.

Example 4.1: Let $N_i = (T^4, \omega_i)$ be a family of symplectic tori, where:

$$\omega = dx_0 \wedge dy_0 + dx_1 \wedge dy_1 + a_i dx_0 \wedge dx_1$$

with $a_i \rightarrow 0$. If we choose $a_i = \frac{n_i}{m_i}$ with $n_i, m_i \rightarrow \infty$, then any non trivial periodic solutions with respect to, say $f(x, y) = \cos(2\pi x_0)$ must have divergent periods as $i \rightarrow \infty$.

In particular the almost Kaehler sequences:

$$M_i = (N_0 \times \cdots \times N_i, \omega_0 + \cdots + \omega_i)$$

certainly admits asymptotic periodic solutions with the period 1, with respect to the bounded Hamiltonian:

$$f(x_0, y_0, x_1, y_1, \dots) = \cos(2\pi x_0)$$

but they do not admit any periodic solutions. Here we embed $M_i \hookrightarrow M_{i+1}$ by $(z_0, \dots, z_i) \rightarrow (z_0, \dots, z_i, 0)$.

4.C Capacity functions over almost Kaehler sequences: In this section we introduce 3 variants of capacity invariants:

$$\text{cap}, \quad \text{As-cap}, \quad \text{C}$$

over almost Kaehler sequences. cap is the most standard one and extends the finite dimensional case directly. As-cap and C both arose by looking at ‘infinitesimal stabilizations’ of cap . We verify that in some case they coincide each other.

4.C.1 Capacity on almost Kaehler sequences: Let $f : M \rightarrow [0, \infty)$ be a bounded Hamiltonian over an almost Kaehler sequence $[(M_i, \omega_i, J_i)]$. We say that f is *pre admissible*, if there exist open sets $U, V \subset M$ with $f|_U \equiv \sup f$ and $f|_V \equiv 0$.

Let us say that f is *admissible*, if it is pre admissible, and for any non trivial asymptotic loops $[x_i]$, if it consists of periodic solutions $x_i : [0, T_i] \rightarrow M_i$ with respect to the restrictions $f_i \equiv f|_{M_i} : M_i \rightarrow \mathbf{R}$ for all $i \geq i_0 \geq 0$, then thier periods satisfy the bounds:

$$T(x_i) = T_i > 1.$$

Let $[(X_i, \omega_i, J_i)]$ be an almost Kaehler sequence with boundary $\partial X = \cup_i \partial X_i$ with $\partial X_i \subset \partial X_{i+1}$. A bounded Hamiltonian $f : \cup_i X_i \mapsto [0, \infty)$ is

pre admissible, if there is an open set $U \subset \cup_i X_i$ with $f|_U \equiv \sup f$, and $f|_V \equiv 0$ holds on $\delta > 0$ neighbourhood V of the boundary $\partial X \subset V$ for some positive $\delta = \delta(f) > 0$. We say that f is *admissible*, if it is pre admissible and satisfies the above property.

Let us put the set of admissible functions by $H_a([(M_i, \omega_i, J_i)])$. Recall $m(f) = \sup f - \inf f \geq 0$.

Definition 4.2 *The capacity function cap is defined by:*

$$cap([(M_i, \omega_i, J_i)]) = \sup\{m(f) : f \in H_a([(M_i, \omega_i, J_i)])\}.$$

cap takes values in $\mathbf{R}_+ \cup \{+\infty\}$.

Proposition 4.2 *Let $[(M_i, \omega_i, J_i)]$ be a minimal almost Kahler sequence with a fixed minimal element $\alpha \in \pi_2(M)$ with $m = \langle \omega, \alpha \rangle$.*

Then the estimates:

$$cap([(M_i, \omega_i, J_i)]) \leq m$$

hold, if the moduli space $\mathfrak{M}([(M_i, \omega_i, J_i)])$ is non empty, regular, has 1 dimensional and S^1 freely cobordant to non zero.

Proof: Let $f : M \rightarrow \mathbf{R}$ be pre admissible, and $f_i : M_i \rightarrow \mathbf{R}$ be its restrictions. It is enough to check that there is a family of periodic solutions $x_i : S^1 \rightarrow M_i$ with the properties:

- (1) their periods satisfy uniform lower bounds $T(x_i) > \delta > 0$ from below,
- (2) they consist of non trivial asymptotic loops with length $x_i > \delta' > 0$.

By lemma 4.1, periodic solutions $x_i : S^1 \rightarrow M_i$ exist certainly. In fact the construction by [HV] with the proof of lemma 2.2 verifies the above two properties for these periodic solutions. We outline the proof below.

Let $B_\mu(p_0), B_\mu(p_\infty) \subset M = \cup_i M_i$ be $\mu > 0$ balls so that $f|_{B_\mu(p_0)} \equiv 0$ and $f|_{B_\mu(p_\infty)} \equiv \sup f$.

Step 1: For $c > 0$, there exist $\epsilon > 0$ so that for any s_0 and solution (λ, u) to the equation $(f_i)_\lambda(u) = 0$ as in [HV] (2.12) on page 599 with:

$$\begin{aligned} |\nabla u(s, t)| &\leq c \quad \text{for } s \in (-\infty, s_0] \times S^1, \\ u((-\infty, s_0] \times S^1) \cap B_\mu(p_0)^c &\neq \emptyset \end{aligned}$$

then the lower estimates hold:

$$\int_{-\infty}^{s_0} \int_0^1 u^*(\omega) \geq \epsilon.$$

The proof goes by the same way as [HV] in lemma 3.1, where we introduce minor modifications of the arguments in order to induce the uniform bounds as above.

Let us consider smooth maps $\gamma : S^1 \rightarrow B_\mu(p_0)$ with the L^2 bounds length $\int_{S^1} |\dot{\gamma}|^2 \leq c$ for a constant $c > 0$. Let us regard $\gamma : S^1 \rightarrow B_\mu(p_0) \subset H$ as maps into the Hilbert space, and consider $p_0 - \gamma : S^1 \rightarrow H$. We claim that for any $\delta > 0$, there is a constant $\tau = \tau(\delta, c)$ so that if $\|p_0 - \gamma\|_{L^2} \leq \tau$, then $\text{diam } \gamma \leq \delta$ hold. The argument in [HV] p608 almost works except that we are concerning maps into infinite dimensional spaces, and so we cannot use the maps themselves directly to apply the Ascoli-Arzelà Theorem. We just modify the argument by replacing $p_0 - \gamma$ by the functions $|p_0 - \gamma| : S^1 \rightarrow [0, \infty)$ by taking the pointwise norms.

Let $\gamma_k : S^1 \rightarrow B_\mu(p_0)$ be a family of smooth maps with the uniform bounds $\int_{S^1} |\dot{\gamma}_k|^2 \leq c$. By the compact embedding $L_1^2(S^1) \hookrightarrow C^0(S^1)$ between the real valued functional spaces, if $|p_0 - \gamma_k| : S^1 \rightarrow [0, \infty)$ converges to zero in L^2 , then they must also converge to zero in C^0 . In particular the diameters of $\{\gamma_k\}_k$ also converge to zero. This is enough to conclude the claim.

Let us introduce the infinite dimensional Poincaré lemma, which states if a closed form on a ball in the Hilbert space satisfy C^0 bounds, then one can find its primitive form which lies in the dual space of L^2 vectors. Let:

$$\omega = \sum_{i>j \geq 0} a_{ij} dx_i \wedge dx_j$$

be a closed form on $B_\mu(p_0)$ with the uniform bounds $\|a_{i,j}\|_{C^0(B_\mu(p_0))} \leq C$. Then the primitive θ with $d\theta = \omega$ is given by the one form:

$$\theta = \sum_{i>j \geq 0} \left(\int_0^1 a_{ij}(t\bar{x}) t dt \right) x_i dx_j$$

which satisfies $d\theta = \omega$ and is clearly L^2 one form.

As a result there is a constant c with the estimates $|\int \gamma^*(\theta)| \leq c \int |\dot{\gamma}|^2$ for any loops $\gamma : S^1 \rightarrow B_\mu(p_0)$ by the Cauchy-Schwartz estimate.

Step 2: There are uniform constants c_α so that any solutions (λ, u) to the equations $(f_i)_\lambda(u) = 0$ satisfy uniform bounds $\|u\|_{C^\alpha(S^2)} \leq c_\alpha$. This is verified by following the argument in [HV] except taking the limit to obtain holomorphic spheres, where instead we use a similar method as the proof of lemma 2.2 particularly step 4.

The rest of the arguments are also parallel again by use of a similar method as step 4 as above. This completes the proof.

4.C.2 Asymptotic capacity: Below we introduce two stable versions of the capacity function. Let $f : M \rightarrow [0, \infty)$ be a bounded Hamiltonian. Let us say that f is *as-admissible*, if it is pre admissible, and any non trivial asymptotic periodic solution has its period:

$$\liminf_i T(x_i) > 1.$$

Similarly we define as-admissibility for bounded Hamiltonians on almost Kaehler sequences with boundary.

Let us put the set of as-admissible functions by $H_{\text{as}}([(M_i, \omega_i, J_i)])$, and we define the *asymptotic capacity*:

$$\text{As-cap}([(M_i, \omega_i, J_i)]) = \text{Sup}\{m(f) : f \in H_{\text{as}}([(M_i, \omega_i, J_i)])\}.$$

As-cap also takes values in $\mathbf{R}_+ \cup \{+\infty\}$. Notice that a-priori estimate:

$$\text{As-cap}([(M_i, \omega_i, J_i)]) \leq \text{cap}([(M_i, \omega_i, J_i)])$$

holds.

4.C.3 Capacity over infinitesimal neighbourhoods: Let us introduce the geometric version of stabilized capacity function which is formulated by use of the infinitesimal neighbourhoods.

Let $[(M_i, \omega_i, J_i)]$ be an almost Kaehler sequence. The *capacity invariant over infinitesimal neighbourhoods* are defined by:

$$C^m([(M_i, \omega, J_i)]) = \sup\{ \limsup_l \text{cap}([(M_i^l, \omega_i^l, J_i^l)]) : \\ [(M_i^l, \omega_i^l, J_i^l)] \in \mathfrak{N}^m([(M_i, \omega_i, J_i)]) \}$$

Notice that a priori estimates hold:

$$\text{As-cap} \leq \text{cap} \leq C^m$$

Theorem 4.1 *Let $[(M_i, \omega, J_i)]$ be a minimal, isotropically symmetric and quasi transitive Kaehler sequence, with a fixed minimal element $\alpha \in \pi_2(M)$.*

If the moduli space of holomorphic curves is non empty, regular, 1 dimensional and S^1 freely cobordant to non zero, then the estimate:

$$C^m([(M_i, \omega_i, J_i)]) \leq m$$

holds, where $m = \langle \omega, \alpha \rangle$.

Proof: Strong regularity holds since it is a regular and symmetric Kaehler sequence. Then $\mathfrak{R}^m([(M_i, \omega_i, J_i)])$ is strongly regular since it is minimal and quasi-transitive. For any $[(M_i^l, \omega_i^l, J_i^l)] \in \mathfrak{R}^m([(M_i, \omega_i, J_i)])$, there is large l_0 so that for all $l \geq l_0$, the moduli spaces over $[(M_i^l, \omega_i^l, J_i^l)]$ are all homeomorphic to the one over $[(M_i, \omega_i, J_i)]$ by proposition 3.1.

Then the estimates $\text{cap}([(M_i^l, \omega_i^l, J_i^l)]) \leq m$ hold by combining with these with lemma 4.1. This completes the proof.

So far we have introduced three versions of the capacity invariants. We would like to propose:

Conjecture 4.1: Let $[(M_i, \omega_i, J_i)]$ be a Kaehler sequence. Under the conditions of theorem 4.1, the equalities hold:

$$\text{As-cap}([(M_i, \omega_i, J_i)]) = \text{cap}([(M_i, \omega_i, J_i)]) = C^m([(M_i, \omega_i, J_i)]).$$

We verify the equalities for both the infinite unit disk and \mathbf{CP}^∞ below.

4.D Examples: Here we estimate the values of capacities over some concrete cases. These include infinite dimensional disks, infinite projective space and infinite tori.

4.D.1 Infinite disks: Let $D^{2i} \subset \mathbf{R}^{2i}$ be the standard unit disks, and $[D^{2i}]$ be the standard Kaehler sequence with $\cup_i D^{2i} \subset \mathbf{R}^{2\infty}$.

Let us fix $N \geq 1$ and put the infinite products of D^{2N} :

$$\begin{aligned} \tilde{D} &= D^{2N} \times D^{2N} \times \dots, \\ D(i) &= D^{2N} \times \dots \times D^{2N} \subset \tilde{D} \quad (i \text{ times}) \end{aligned}$$

where we embed $D(i) \subset D(i+1)$ by $(m_1, \dots, m_i) \rightarrow (m_1, \dots, m_i, 0)$.

Here we verify the following equalities:

Proposition 4.3 *Let Cap be $As\text{-}cap$ or cap . Then the equalities hold:*

$$C^m([\mathbf{CP}^i]) = Cap([\mathbf{CP}^i]) = Cap([D^{2i}]) = \pi.$$

Proof: We split the proof into three cases:

$$(1) \pi \geq C^m([\mathbf{CP}^i]), \quad (2) \text{As-cap}(\mathbf{CP}^\infty) \geq \text{As-cap}([D^{2i}]) \geq \pi.$$

The proof of (2) is postponed to the following lemmas below.

Let us consider (1). \mathbf{CP}^∞ is minimal with $m = \pi$, since it has π_2 -rank 1 and the embedding $\mathbf{CP}^1 \subset \mathbf{CP}^\infty$ is holomorphic. It is isotropically symmetric by Examples 1.4 and quasi transitive by lemma 1.7.

It is strongly regular by example 2.2, whose moduli space is homeomorphic to S^1 which is S^1 freely cobordant to non zero.

So all the conditions in theorem 4.1 are satisfied, and so (1) follows.

Lemma 4.2 *$As\text{-}cap([D(i)]) \geq As\text{-}cap([D^{2i}]) \geq \pi$.*

Proof: The first inequality is clear, since there are compatible embeddings $D^{2iN} \hookrightarrow D(i)$. So we will consider the second one.

We follow a parallel argument to [HZ] p72. Here one needs to be careful about treating asymptotic solutions.

In [HZ], $c(D^{2i}) = \pi$ is verified for any i . In fact the following is shown; let $f : [0, 1] \mapsto [0, \pi]$ be any function satisfying:

- (1) $0 \leq -f' < \pi - \delta$, $\delta > 0$,
- (2) $f(t) = \pi - \delta'$, $t \sim 0$, $\delta' > 0$ and $f(t) \equiv 0$, $t \sim 1$.

Then $F : D^{2i} \mapsto [0, \pi]$ gives an admissible function on D^{2i} by $F(x) = f(|x|^2)$. Let us put $F : \cup_i D^{2i} \mapsto [0, \pi]$ by the same way, which is a bounded Hamiltonian. We claim that F is as-admissible in our sense. Let us take any non trivial asymptotic periodic solution $[x_i]$. By the definition, this family satisfies:

$$|J\dot{x}_i + \nabla F(x_i)|^{C^0} = |J\dot{x}_i + 2f'(|x_i|^2)x_i|^{C^0} \rightarrow 0.$$

Let us put $G(x) = \frac{1}{2}|x|^2$. Then one has the uniform estimate:

$$\frac{d}{dt}G(x_i(t)) = \langle \nabla G, \dot{x}_i \rangle \sim \langle \nabla G, J\nabla F \rangle (x_i(t)) = 0.$$

Thus there is a constant $0 < a \leq 2(\pi - \delta)$ with $\sup_t |2f'(|x_i(t)|^2) + a| \rightarrow 0$.

Let us put $a_i = -2f'(|x_i(0)|^2)$ with $a_i \rightarrow a$. For $y_i \equiv \exp(-a_i Jt)x_i(0)$, the family $[y_i]$ consists of a periodic solution. Then $\sup_t |y_i(t) - x_i(t)|C^0 \rightarrow 0$ hold since the Hamiltonian vector field is uniformly Lipschitz (of completely bounded geometry). It also implies $\sup_t |y_i(t) - x_i(t)|C^1 \rightarrow 0$, since they are (asymptotic) solutions. So $\lim_i T(x_i) > 1$ follows since $T(y_i) > 1 + \delta'$ hold for some $\delta' > 0$. This completes the proof.

Lemma 4.3 $As\text{-}cap(\mathbf{CP}^\infty) \geq As\text{-}cap([D^{2i}])$.

Proof : There are standard symplectic embeddings $D^{2i} \hookrightarrow \mathbf{CP}^i$ from the unit disks. Let J_1 be the induced almost complex structure, which is different from the standard one J_0 on D^{2i} . Thus the induced Riemannian metrics on D^{2i} are not standard. Recall that as-admissibility of bounded Hamiltonians involves asymptotic behaviour of loops by the Riemannian norms. However this does not cause any problem for us, since we use bounded Hamiltonians over $\cup_i D^{2i}$ which vanish near boundary, and so the induced Riemannian metrics are uniformly equivalent to the standard one on the support of f in the following sense. Let us choose any pre-admissible Hamiltonian f on $\cup_i D^{2i}$ with respect to J_1 . One may assume that for any small $\mu > 0$ and any non trivial asymptotic periodic solution $[x_i]$, there is a large i_0 so that $\text{Supp } x_i$ are all contained in μ neighbourhoods of $\text{Supp } f_i$ with $f_i = f|_{D^{2i}}$ for all $i \geq i_0$. Then there is an equivalence of the induced Riemannian metrics, $C^{-1}| \cdot |_0 \leq | \cdot |_1 \leq C| \cdot |_0$, where C depends only on $d(\partial \cup_i D^{2i}, \text{Supp } f)$. Thus if $[x_i]$ is an asymptotic periodic solution for J_0 , then it is also the case for J_1 . The converse is the same. Thus the inequality $As\text{-}cap(\mathbf{CP}^\infty) \geq As\text{-}cap([D^{2i}]) \geq \pi$ follows.

This completes the proof.

Let us see how the capacity invariant behave under small perturbation of the forms. Notice that the argument below works only for the products of disks. The situation is completely different for the torus case.

Sublemma 4.1 *Let us fix i and let ω' be a symplectic form on $D(i)$ which is sufficiently near the standard form ω . Then the values of capacity*

$c(D(i), \omega') \sim c(D(i), \omega)$ are sufficiently near from each other.

Proof : This follows from the Darboux-chart construction. Let us denote $D_i(\delta) = D^{2N}(\delta) \times \cdots \times D^{2N}(\delta)$, where $D^{2N}(\delta)$ is the $2N$ dimensional δ disk. It is enough to find small $\epsilon > 0$ with $\epsilon \rightarrow 0$ as $\omega' \rightarrow \omega$, and symplectic embeddings:

$$I : (D_i(1 - \epsilon), \omega) \hookrightarrow (D(i), \omega'), \quad I' : (D_i(1 - \epsilon), \omega') \hookrightarrow (D(i), \omega).$$

The constructions of these are parallel, and we will check the first case only.

Let us find a diffeomorphism $\varphi : D \equiv D_i(1 - \epsilon) \hookrightarrow D(i)$ with $\varphi^*(\omega') = \omega$. Let us put $\omega_t = \omega + t(\omega' - \omega)$ for $t \in [0, 1]$. By the assumption, all ω_t give symplectic forms. We will find a smooth family of diffeomorphisms $\varphi_t : D \hookrightarrow D(i)$ given as the flow of vector field X_t , and satisfying $\varphi_t^*(\omega_t) = \omega$. The last equality is equivalent to $d(i_{X_t}\omega_t) + \omega - \omega' = 0$.

By the Poincaré lemma, there is a smooth one form λ on $D(i)$ with $\omega - \omega' = d\lambda$ and $\lambda(0) = 0$. The pointwise norm of λ will be sufficiently small. Then we choose unique X_t satisfying $\omega_t(X_t, \cdot) = -\lambda$ and $X_t(0) = 0$. Thus $|X_t|$ is also small pointwisely. In particular the flow of X_t exists for all $0 \leq t \leq 1$ with image in $D(i)$, if we start from any point $p \in D = D_i(1 - \epsilon)$. Thus we have obtained the desired φ_t . This completes the proof.

4.D.2 Based admissibility: Let us equip the standard metric on $\tilde{D} \equiv \cup_i D(i) \subset \mathbf{R}^\infty$, and let $f : \tilde{D} \mapsto [0, \infty)$ be a smooth and bounded function. We say that a non trivial asymptotic periodic solution $[x_i]$ is *based*, if $\{|x_i|C^0(S^1)\}$ is uniformly bounded, and there is some $i_0 \geq 1$ so that $\liminf_i \text{length } P_{i_0}(x_i) > 0$, where:

$$P_{i_0} : \tilde{D} = D^{2N} \times D^{2N} \times \cdots \mapsto D(i_0) = D^{2N} \times \cdots D^{2N} \quad (i_0 \text{ times})$$

are the projections. We will say that f is *based admissible*, if any based asymptotic periodic solutions $[x_i]$ satisfy $\liminf_i T_i > 1$.

Let us construct based admissible functions by taking infinite products of some $f : D^{2N} \mapsto [0, 1]$. Let us choose a smooth function $g : [0, 1] \rightarrow [0, 1]$ such that for some positive small $\delta, \epsilon > 0$:

$$(1) \quad 0 \leq -g' < 1 + \delta, \quad (2) \quad g|_{[0, \epsilon]} \equiv 1, \quad g|_{[1 - \epsilon, 1]} \equiv 0.$$

Then we put $f : D^{2N} \mapsto [0, 1]$ by $f(p) = g(|p|^2)$. f is admissible by the proof of lemma 4.2. For convenience, let us assign index on each term:

$$\tilde{D} = D^{2N} \times D^{2N} \times \dots = D_1^{2N} \times D_2^{2N} \times \dots \times D_l^{2N} \times \dots$$

We also denote f_1, f_2, \dots where all f_i are the same f and we regard f_i as functions on D_i^{2N} . Let us put:

$$\begin{aligned} F &= f_1 f_2 \dots f_l \dots : \tilde{D} \mapsto [0, 1], \\ F_i &= f_1 \dots f_i : D(i) = D_1^{2N} \times \dots \times D_i^{2N} \mapsto [0, 1] \end{aligned}$$

where $f_i f_j$ imply the pointwise multiplications. Notice $F(0) = 1$ at $0 \in \tilde{D}$. We claim that F is based admissible. In fact, the Hamilton vector field is:

$$\begin{aligned} X_F &= f_2 f_3 \dots X_{f_1} + f_1 f_3 f_4 \dots X_{f_2} + f_1 f_2 f_4 f_5 \dots X_{f_3} + \dots \\ &\equiv G_1 X_{f_1} + G_2 X_{f_2} + \dots \end{aligned}$$

G_i take the values in $[0, 1]$.

Let us take a based asymptotic periodic solution $[l(i)]$ and denote its period $T = \liminf_i T(l(i)) > 0$. Let us fix i_0 , and denote by l_i as the projection of $l(i)$ on $D_{i_0}^{2N}$ component. Then both the convergences:

$$\lim_i |\dot{l}_i - G_{i_0} X_{f_{i_0}}|_{C^0} = 0, \quad \lim_i |l(i) - X_F|_{C^0} = 0 \quad (*)$$

hold. One may assume $\liminf_i \text{length } l_i > 0$.

The following shows F is based admissible.

Sublemma 4.2 *The periods $T(l_i)$ of l_i are larger than 1 for large i .*

Proof : We will use the properties that each f_i is admissible on D_i^{2N} and G_i takes values less than 1. By choosing a subsequence, the family $\{l_i\}_i$ converges to l_∞ in $C^0(D_{i_0}^{2N})$. Let us put $g_i = G_{i_0}(l(i)(t)) : [0, T(l(i))] \mapsto [0, 1]$. Then also a subsequence of $\{g_i\}_i$ converges to $g : [0, T] \mapsto [0, 1]$ in C^0 by (*). Thus $\{l_i\}_i$ converges in fact in $C^1(D_{i_0}^{2N})$ topology.

Now the equality $\dot{l}_\infty(t) = g(t)X_{f_{i_0}}$ holds. Then there is a small $\delta > 0$ so that $g(t) \geq \delta$ hold, since l_∞ is non trivial. Let us regard it as the periodic function $g : \mathbb{R} \rightarrow (0, \infty)$.

Let us solve the ODE $\alpha : [0, T] \mapsto \mathbb{R}$ satisfying $\dot{\alpha}(t) = g^{-1}(\alpha(t))$ with $\alpha(0) = 0$. Then put $l' : [0, T] \mapsto D_{i_0}^{2N}$ by $l'(t) = l_\infty(\alpha(t))$, which satisfies the equation $\dot{l}'(t) = g^{-1}(\alpha(t))\dot{l}_\infty(\alpha(t)) = X_{f_{i_0}}(l'(t))$.

The image of l' consists of the loop, and there is some $0 < T' \leq T$ with $l'(0) = l'(T')$, since $\dot{\alpha}(t) \geq 1$ for all t . Since f_{i_0} is admissible, it follows $T \geq T' > 1$. This completes the proof.

4.D.3 Infinite tori: Let $T^{4\infty} = [(T^{4n}, \omega, J)]$ be the standard tori. We verify the following:

Proposition 4.4 $C([(T^{4n}, \omega, J)]) = \infty$

Proof : Let us consider the standard torus embeddings:

$$T^4 \subset T^8 \subset T^{12} \subset \dots \subset T^{4n} \subset \dots$$

Let us choose small real numbers $\alpha_1, \alpha_2 \in \mathbf{R}$. Then one has a family of symplectic structures on T^4 by:

$$\omega(\alpha_1, \alpha_2) = dx_2 \wedge dx_1 + dx_4 \wedge dx_3 + \alpha_1 dx_3 \wedge dx_2 + \alpha_2 dx_1 \wedge dx_3.$$

Sublemma 4.3 (Z, see HZ) *Suppose $(\alpha_1, \alpha_2) \in \mathbf{R}^2$ is irrational. Then the corresponding symplectic manifold $(T^4, \omega(\alpha_1, \alpha_2))$ has infinite capacity.*

Proof of lemma: Let us choose α_i as above, put the corresponding form by $\omega(\alpha_i) = \omega(\alpha_1, \alpha_2)$ and $\omega = \omega(0, 0)$ be the standard form on T^4 . Then we have a symplectic sequence:

$$\begin{aligned} T^{4\infty}(\alpha_1, \alpha_2) &= (T^4, \omega(\alpha_1, \alpha_2)) \times (T^4, \omega) \times (T^4, \omega) \times \dots \\ T^{4n}(\alpha_1, \alpha_2) &= (T^4, \omega(\alpha_1, \alpha_2)) \times (T^4, \omega) \times \dots \times (T^4, \omega) \quad (n \text{ times}). \end{aligned}$$

Let (T^4, ω, J) be the standard Kaehler structure. Since $\omega(\alpha_1, \alpha_2)(V, JV) > 0$ take positive values for $V \neq 0$ (α_i are small), one can construct another J' which is compatible with $\omega(\alpha_1, \alpha_2)$ and is near J . Thus $(T^4, \omega(\alpha_i), J')$ gives another almost Kaehler data which is sufficiently near the standard.

Let us choose families of pairs (α_1^l, α_2^l) and the compatible almost complex structures J^l as above so that $\alpha_i^l \rightarrow 0$ and $J^l \rightarrow J$ as $l \rightarrow \infty$. Then we have a convergent family of product almost Kaehler sequences

$\{[T^{4n}, \omega(\alpha_1^l, \omega_2^l), J^l]\}_l$ where we choose smooth embeddings $T^{4n} \hookrightarrow T^{4n}$ by the identity for every n . So the family gives an element:

$$[[T^{4n}, \omega(\alpha_1^l, \omega_2^l), J^l]] \in \mathfrak{N}([T^{4n}, \omega, J]).$$

Thus it is enough to see $\text{cap}([T^{4n}, \omega(\alpha_1^l, \omega_2^l), J^l]) = \infty$. By sublemma 4.3, there are admissible functions $f : (T^4, \omega(\alpha_1^l, \omega_2^l)) \mapsto \mathbf{R}$ with sufficiently large $m(f)$. f induce the bounded Hamiltonians on $[T^{4n}, \omega(\alpha_1^l, \omega_2^l), J^l]$, which are also admissible over $[T^{4n}, \omega(\alpha_1^l, \omega_2^l), J^l]$. This implies that the capacity is infinite. This completes the proof.

5 Completion of spaces

In this section we extend the class of the maps we treat to L^2 loops over almost Kaehler sequences.

5.A Completion of spaces: Let $[(M_i, \omega_i, J_i)]$ be an almost Kaehler sequence, and take two ϵ complete almost Kaehler charts:

$$\varphi(p_i) : D(\epsilon) \cong D_{p_i} \subset M \cup_{j \geq 0} M_j$$

for $i = 1, 2$ and $D(\epsilon) \subset \mathbb{R}^\infty$. Let us denote the completion of $D(\epsilon)$ by $\bar{D}(\epsilon) \subset H$ where H is the Hilbert space obtained by completion of \mathbb{R}^∞ with the standard metric.

Let us introduce an equivalent relation as follows. Two points $z_1, z_2 \in \bar{D}(\epsilon)$ are equivalent, if there is a sequence:

$$\{m_1, m_2, \dots\} \subset D_{p_1} \cap D_{p_2} \subset M$$

so that the corresponding two sequences:

$$w_k^i \equiv \varphi(p_i)^{-1}(m_k) \subset \bar{D}(\epsilon)$$

converge to z_1 and z_2 in $\bar{D}(\epsilon)$ respectively.

Definition 5.1 *The completion of $M = \cup_i M_i$ is defined by:*

$$M \subset \bar{M} \equiv \coprod \bar{D}(\epsilon) / \sim .$$

The set of points at infinity consists of:

$$\partial M \equiv \bar{M} \setminus M.$$

Lemma 5.1 *Suppose $[(M_i, \omega_i, J_i)]$ is an almost Kaehler sequence. Then \bar{M} admits the structure of a Hilbert manifold.*

Proof: This follows from lemma 1.8. This completes the proof.

Notice that by definition, any bounded Hamiltonian $f : M \rightarrow \mathbf{R}$ extends to a smooth and bounded function $f : \bar{M} \rightarrow \mathbf{R}$.

Later on we assume that \bar{M} admit the Hilbert-manifold structure. Let $U_\epsilon(M_k) \subset M$ be the ϵ neighbourhoods of M_k . We denote their completions by $\bar{U}_\epsilon(M_k) \subset \bar{M}$. Then the holomorphic maps are extended as:

$$\pi_k : \bar{U}_\epsilon(M_k) \rightarrow M_k.$$

5.B L^2 holomorphic curves: Let us introduce the Sobolev spaces of maps into the completed spaces. Notice that in order to define the full Sobolev spaces, one has to use cut off functions over the local charts, which play the role of the locally finite partition of unities in the finite dimensional setting. In our situation the Sacks-Uhlenbeck's estimates allow us to obtain uniform norms which is necessary to perform analysis of the moduli spaces we study.

Let us take small $\delta > 0$ and choose $\frac{1}{2}\delta$ net which is consisted by finite set of points $\{x_1, \dots, x_m\} \subset S^2$. Let $B_i(\delta) \subset S^2$ be δ balls with the center x_i , so that $\{B_1(\frac{\delta}{2}), \dots, B_m(\frac{\delta}{2})\}$ consists of an open covering of S^2 . Let us choose a smooth function $\xi_i : B_i(\delta) \rightarrow [0, 1]$ and have a partition of unity by:

$$\rho_i(x) = \frac{\xi_i(x)}{\sum_{i=1}^m \xi_i(x)} \quad (x \in S^2), \quad \xi_i(x) = \begin{cases} 0 & x \in \partial B_i(\delta) \\ 1 & x \in B_i(\frac{\delta}{2}) \end{cases}$$

Let $\varphi_i : \bar{D}(\epsilon) \cong \bar{D}_{p_i} \subset \bar{M}$ be complete almost Kaehler charts for finite sets $\{p_1, \dots, p_m\} \subset M$. Let $u_0 : S^2 \rightarrow \bar{M}$ be a map with:

$$u_0(B_i(\delta)) \subset \bar{D}_{p_i}$$

such that $\rho_i u_0 \in L_{l+1}^2(B_i(\delta) : \bar{D}_\epsilon)$. Then we obtain the function spaces around u_0 defined by:

$$L_{l+1}^2(S^2, \bar{M}; \{p_i\}_i) = \{u : S^2 \rightarrow \bar{M} : u(B_i(\delta)) \subset \bar{D}_{p_i} \rho_i u \in L_{l+1}^2(B_i(\delta) : \bar{D}_\epsilon)\}$$

where we equip with the norms:

$$\|u\|_{L_{l+1}^2}^2 = \sum_{i=1}^l \|\rho_i(u)\|_{L_{l+1}^2}^2.$$

These spaces admit the Hilbert manifold structure on small neighbourhoods of u_0 .

Remark 5.1: One will be able to obtain the globally defined function spaces as Hilbert manifolds by use of the infinite dimensional version of the Levi-Civita connection ([Kl]).

With the fixed data $\{x_i\}_i$, we define the function spaces:

$$L_{l+1}^2(S^2, \bar{M}) \equiv \cup_{\{p_i\}_{i=1}^l \subset M} L_{l+1}^2(S^2, \bar{M}; \{p_i\}_i).$$

Let us introduce the functional spaces which are parallel to section 2. We define the spaces of Sobolev maps:

$$\begin{aligned} \mathfrak{B} &= \{u \in L_{l+1}^2(S^2, \bar{M}) : [u] = \alpha, \\ &\int_{D(1)} u^*(\omega) = \frac{1}{2} \langle \omega, \alpha \rangle, \quad u(*) = p_* \in M_0, * \in \{0, \infty\}\}. \end{aligned}$$

Let $E(J)$, $F \mapsto S^2 \times \bar{M}$ be vector bundles whose fibers are respectively:

$$\begin{aligned} E(J)(z, m) &= \{\phi : T_z S^2 \mapsto T_m \bar{M} : \text{anti complex linear map}\}, \\ F(z, m) &= \{\phi : T_z S^2 \mapsto T_m \bar{M} : \text{linear}\}. \end{aligned}$$

Then we have the Hilbert bundle and the Cauchy-Riemann operators:

$$\begin{aligned} \mathfrak{E} &= L_l^2(\mathfrak{B}^*(E(J))) = \cup_{u \in \mathfrak{B}} \{u\} \times L_l^2(u^*(E(J))), \\ \bar{\partial}_J &\in C^\infty(\mathfrak{E} \mapsto \mathfrak{B}). \end{aligned}$$

Lemma 5.2 *There is m determined only by $[(M_i, \omega_i, J_i)]$, and for any l there is $\epsilon = \epsilon(l) > 0$ so that ϵ neighbourhoods of any holomorphic curves $u : S^2 \rightarrow \bar{M}$ with $u(0) = p_0$ and $u(\infty) = p_\infty$, admit the Hilbert manifold structures in $L_{l+1}^2(S^2, \bar{M}; \{p_i\}_{i=1}^m)$ where $p_i = u(x_i)$.*

Proof: This follows from lemma 2.2 since the restrictions satisfy $u : B_i(\delta) \rightarrow D_{p_i}$ if $\delta > 0$ is sufficiently but uniformly small. This completes the proof.

Now let us define the moduli space of holomorphic curves by:

$$\bar{\mathfrak{M}}(\alpha, M, J) = \{u \in C^\infty(S^2, \bar{M}) \cap \mathfrak{B} : \bar{\partial}_J(u) = 0\}.$$

A priori inclusions hold:

$$\mathfrak{M}(\alpha, M, J) \subset \hat{\mathfrak{M}}(\alpha, M, J) \subset \bar{\mathfrak{M}}(\alpha, M, J).$$

As before let us say that J is *regular*, if the linealizations $D\bar{\partial}_J(u) : T_u\mathfrak{B} \mapsto \mathfrak{E}_u$ are onto for all $u \in \bar{\mathfrak{M}}$.

Proposition 5.1 *Let $[(M_i, \omega_i, J_i)]$ be a minimal and isotropic symmetric Kaehler sequence.*

If the moduli space of holomorphic curves is non empty, regular and has S^1 freely 1 dimension, then the equality holds:

$$\bar{\mathfrak{M}}(\alpha, M, J) = \mathfrak{M}(\alpha, M, J).$$

Proof: It follows from lemma 2.2 that for any $u \in \bar{\mathfrak{M}}(\alpha, M, J)$, there is some k so that the image of u lies in ϵ neighbourhood of M_k .

Let $\pi_k : \bar{U}_\epsilon(M_k) \rightarrow M_k$ be the holomorphic projections. Then $u_k \equiv \pi_k \circ u : S^2 \rightarrow M_k$ are also holomorphic curves. By proposition 2.2, the moduli spaces are strongly regular at u_k . Since $\mathfrak{M}(\alpha, M, J)$ is compact by lemma 2.8, u_k must be the unique holomorphic curve mod S^1 action, in small neighbourhoods of u_k . So $u = u_k$ must hold.

This completes the proof.

5.C Hamiltonian diffeomorphisms: Let $[(M_i, \omega_i, J_i)]$ be an almost Kaehler sequence, and take a bounded Hamiltonian $f : M = \cup_i M_i \rightarrow \mathbf{R}$.

Let us choose a uniformly bounded covering $\{\varphi(p) : D(\epsilon) \equiv \cup_i D^{2i}(\epsilon) \hookrightarrow \cup_i M_i\}_{p \in M}$. By pulling back as $\varphi(p)^*(f) : D(\epsilon) \rightarrow \mathbf{R}$, let us regard the restriction of the differential df as the one form over $D(\epsilon)$.

The Hamiltonian vector field X_f over $D(\epsilon)$ is defined as the unique vector field which obey the equality:

$$df(Y) = \omega(X_f, Y)$$

for any vector field Y of completely bounded geometry over $D(\epsilon)$. It follows from lemma 1.8 that X_f is in fact determined globally over $M = \cup_i M_i$. The following holds by applying the existence and uniqueness of ODE with infinite dimensional targets:

Lemma 5.3 *Let $[(M_i, \omega_i, J_i)]$ be an almost Kaehler sequence, and X_f be the globally defined vector field over M as above. Then:*

(1) *There is the parametrized diffeomorphisms as its integral:*

$$D_t : \bar{M} \cong \bar{M}.$$

(2) *Let $U \subset \bar{M}$ be an open subset, and $f : D_t(U) \rightarrow \mathbf{R}$ be pre admissible. Then the induced function $D_t^*(f) : U \rightarrow \mathbf{R}$ is also the case.*

We call $D = D_1$ as the *Hamiltonian diffeomorphisms*.

Proposition 5.2 *Let $[(M_i, \omega_i, J_i)]$ be an almost Kaehler sequence. Let us take a bounded Hamiltonian $f : \cup_i M_i \rightarrow \mathbf{R}$ with the Hamiltonian diffeomorphisms $D : \bar{M} \cong \bar{M}$.*

Let $U \subset \bar{M}$ be an open subset. Then we have the equality:

$$As-cap(U) = As-cap(D(U)).$$

Proof: Let $f : D(U) \rightarrow [0, \infty)$ be an as-admissible function, and consider the pull back $D^*(f) : U \rightarrow [0, \infty)$.

Let X_f and $X_{D^*(f)}$ be the Hamiltonian vector fields over $D(U)$ and U respectively. Then the push forward satisfies the equality:

$$D_*(X_{D^*(f)})(m) = X_f(D(m))$$

for $m \in U$.

Let $x_i : [0, T_i] \rightarrow U$ be a non trivial asymptotic periodic solution to $D^*(f)$. It follows from the above equality that $D(x_i) : [0, T_i] \rightarrow D(U)$ must satisfy:

$$\sup_t |D(\dot{x}_i) - X_f(D(x_i))|(t) \rightarrow 0$$

as $i \rightarrow \infty$.

Notice that the images of $D(x_i)$ are certainly contained in \bar{M} , but may not in $\cup_i M_i$. Let $\pi_k : U_\epsilon(M_k) \rightarrow M_k$ be the family of holomorphic projections in definition 1.3.

For each i , there are some k_i so that the followings hold, since $\cup_i M_i$ is dense in \bar{M} :

- (1) the image of $D(x_i)$ are contained in $U_\epsilon(M_{k_i})$,
- (2) $\sup_t |D(x_i) - \pi_{k_i}(D(x_i))| \rightarrow 0$ as $i \rightarrow \infty$.

One may assume $k_i < k_{i+1}$ for all i . Let us put another family of non trivial asymptotic loop:

$$y_l = \pi_{k_i}(D(x_i))$$

for $k_i \leq l \leq k_{i+1} - 1$. Then $y_l : [0, S_l] \rightarrow M_l$ is also an asymptotic periodic solution to f , and so $\liminf T_i = \liminf_l S_l > 1$ must hold by as-admissibility of f . So $D^*(f)$ is also admissible.

In particular we have the inequality $\text{As-cap}(U) \geq \text{As-cap}(D(U))$. The same argument gives us the converse inequality. So we must have the equality. This completes the proof.

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